Area of Existence of Involute Gears

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Management Summary

This paper presents a unique approach and methodology to define the limits of selection for gear parameters. The area within those limits is called the "area of existence of involute gears" (Ref. 1). This paper presents the definition and construction of areas of existence of both external and internal gears. The isograms of the constant operating pressure angles, contact ratios and the maximum mesh efficiency (minimum sliding) isograms, as well as the interference isograms and other parameters are defined. An area of existence allows the location of gear pairs with certain characteristics. Its practical purpose is to define the gear pair parameters that satisfy specific performance requirements before detailed design and calculations. An area of existence of gears with asymmetric teeth is also considered.

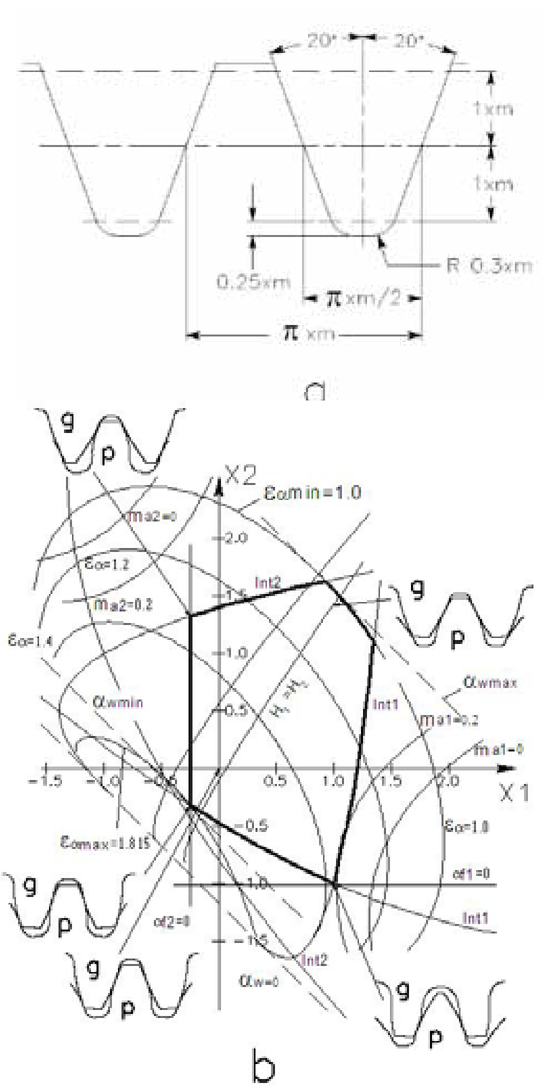

Figure 1a—Standard 20° pressure angle generating rack; b: its block-contour for the pair of gears with number of teeth n_1 = 22 and n_2 = 35.

Introduction

In traditional gear design, the pre-selected basic or generating rack's parameters and its X-shift define the nominal, involute gear geometry. The X-shift selection for the given pair of gears is limited by the block-contour (Refs. 2–3). Borders of the block-contour (Fig.1) include the undercut isograms, the tooth-tip interference isograms, the minimum contact ratio (equal to 1.0 for spur gears) isogram and the isograms of the minimum tooth tip thickness to exclude the gears with the pointed tooth tips. Each point of the block-contour presents the gear pair with a certain set of parameters and performance. If the basic or generating rack parameters (pressure angle, addendum or whole depth) are changed, the block-contour borders will be changed accordingly and will include the gear pair parameters' combinations, which previously could not be achieved yet could present the optimal solution for a particular gear application.

Area of Existence for Symmetric Gearing

The Direct Gear Design method (Refs. 4–5) does not use a pre-selected basic or generating rack to define the gear geometry. Two involutes of the base circle—the arc distance between them and the tooth tip circle— describe the gear tooth (Fig. 2). The equally spaced teeth form the gear. The fillet between the teeth is not in contact with the mating gear teeth, but this portion of the tooth profile is critical because it is the area of the maximum bending stress concentration.

In Direct Gear Design, the selection of parameters for the given gear pair is limited by the area of existence, which was introduced by Prof. E. B. Vulgakov in his "Theory of Generalized Parameters" (Ref. 1). The angles v_1 and v_2 are used as a coordinate system for the area of existence of the involute gear pair with number of teeth n_1 and n_2. The involute profile angles at the tooth tip diameters $\alpha_{a1,2}$ of the mating gears also can be used as a coordinate system for the area of existence. They are:

$$\alpha_{a1,2} = \arccos \frac{d_{b1,2}}{d_{a1,2}} \quad (1)$$

An area of existence is built for the gear pairs with number of teeth n_1 and n_2, and for pre-selected relative tooth thicknesses at the gear tooth tip diameters $m_{a1,2}$. This guarantees avoiding the pointed gear tooth tips and makes the area of existence independent of the gear size. In the metric system, $m_{a1,2} = S_{a1,2}$ / m, where m is operating module in mm. In the English system, $m_{a1,2} = S_{a1,2}$ x DP, where DP is the operating diametral pitch in 1/in. Typically, thicknesses $m_{a1,2}$ are in the range of 0.1– 0.5.

The relation between the involute profile angles ν and α_a is described by equations:

for gears with the external teeth:

$$inv(\nu_{1,2}) - inv(\alpha_{a1,2}) = \frac{S_{a1,2}}{d_{a1,2}} = \frac{m_{a1,2} \times \cos\alpha_{a1,2}}{n_{1,2} \times \cos\alpha_w} \quad (2)$$

for the gear with the internal teeth:

$$\pi / n_2 - inv(\nu_2) + inv(\alpha_{a2}) = \frac{S_{a2}}{d_{a2}} = \frac{m_{a2} \times \cos\alpha_{a2}}{n_2 \times \cos\alpha_w} \quad (3)$$

where: inv(x) = tan(x) - x – involute function.

An area of existence presents a number of isograms that describe gear pairs with certain characteristics, such as the constant operating pressure angle, contact ratio, interference condition or maximum mesh efficiency, etc.

The pressure angle α_w = const isogram equations are (Ref. 1):

for the external gearing:

$$\frac{1}{1+u}(inv(\nu_1) + u \times inv(\nu_2) - \frac{\pi}{n_1}) = inv(\alpha_w) \quad (4)$$

for the internal gearing:

$$\frac{1}{u-1}(u \times inv(\nu_2) - inv(\nu_1)) = inv(\alpha_w) \quad (5)$$

where: $u = n_2/n_1$ – gear ratio.

The contact ratio ε_α= const isogram equation is:

for external gearing:

$$\frac{n_1}{2\times\pi}(\tan\alpha_{a1} + u \times \tan\alpha_{a2} - (1+u)\times\tan\alpha_w) = \varepsilon_\alpha \quad (6)$$

for internal gearing:

$$\frac{n_1}{2\times\pi}(\tan\alpha_{a1} - u \times \tan\alpha_{a2} + (u-1)\times\tan\alpha_w) = \varepsilon_\alpha \quad (7)$$

The external gearing interference isogram equations are:

for the pinion root undercut beginning ($\alpha_{p1} = 0$):

$$\tan((1+u)\times\tan\alpha_w - u\times\tan\alpha_{a2}) = \tan\alpha_{p1} = 0 \quad (8)$$

for the gear root undercut beginning ($\alpha_{p2} = 0$):

$$\tan(\frac{1+u}{u}\times\tan\alpha_w - \frac{1}{u}\times\tan\alpha_{a1}) = \tan\alpha_{p2} = 0 \quad (9)$$

For internal gearing, interference when the pinion root undercut beginning (α_{p1}= 0) is:

$$\tan(u\times\tan\alpha_{a2} - (u-1)\times\tan\alpha_w) = \tan\alpha_{p1} = 0 \quad (10)$$

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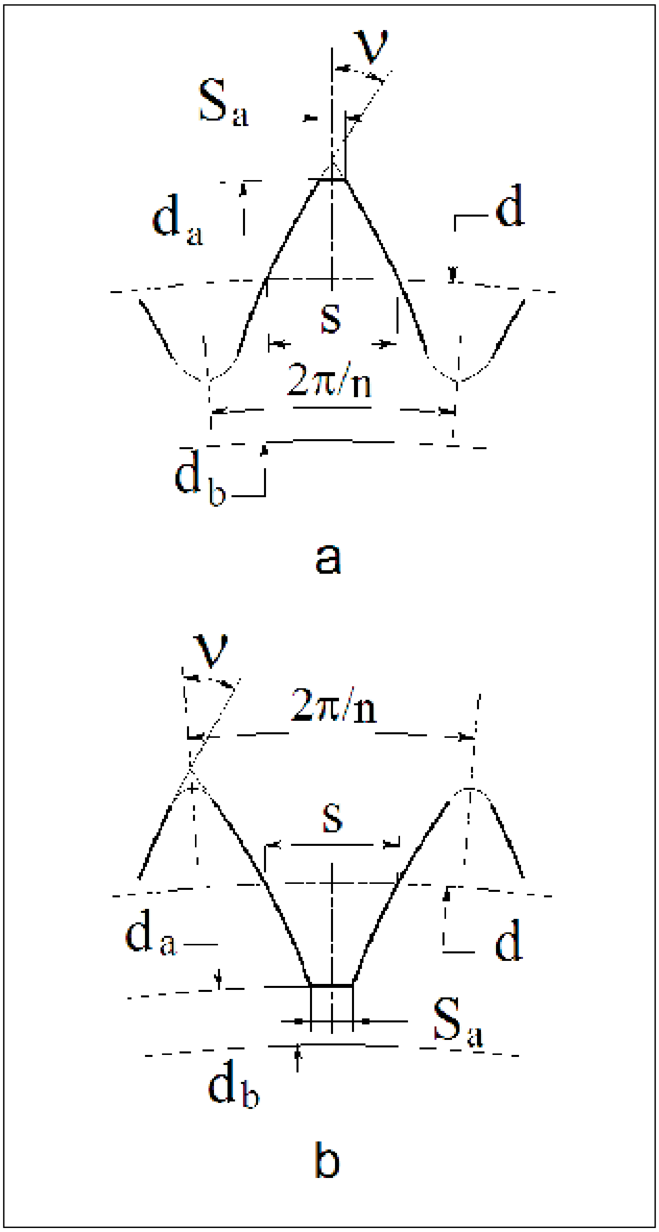

Figure 2—Tooth profile. a: external tooth; b: internal tooth; n: number of teeth; d_a: tooth tip circle diameter; d_b : base circle diameter; d: reference circle diameter; S: circular tooth thickness at the reference diameter; v: involute intersection profile angle; S_a: circular tooth thickness at the tooth tip diameter.

For the gear with internal teeth, the root undercut does not exist. However, there is another "tip-tip" interference possibility in internal gearing. Its equation is:

$$\lambda_1 - u \times \lambda_2 = 0 \qquad (11)$$

where angles:

$$\lambda_{1,2} = \gamma_{1,2} + inv(\alpha_{a1,2}) - inv(\alpha_w) \qquad (12)$$

$$\gamma_1 = \pi - \arccos\left(\frac{\left(\frac{n_1 \times \cos\alpha_w}{\cos\alpha_{a1}}\right)^2 + (n_1+n_2)^2 - \left(\frac{n_2 \times \cos\alpha_w}{\cos\alpha_{a2}}\right)^2}{\frac{n_1 \times \cos\alpha_w}{\cos\alpha_{a1}} \times (n_1+n_2)}\right) \qquad (13)$$

$$\gamma_2 = \arccos\left(\frac{\left(\frac{n_2 \times \cos\alpha_w}{\cos\alpha_{a2}}\right)^2 + (n_1+n_2)^2 - \left(\frac{n_1 \times \cos\alpha_w}{\cos\alpha_{a1}}\right)^2}{\frac{n_2 \times \cos\alpha_w}{\cos\alpha_{a2}} \times (n_1+n_2)}\right) \qquad (14)$$

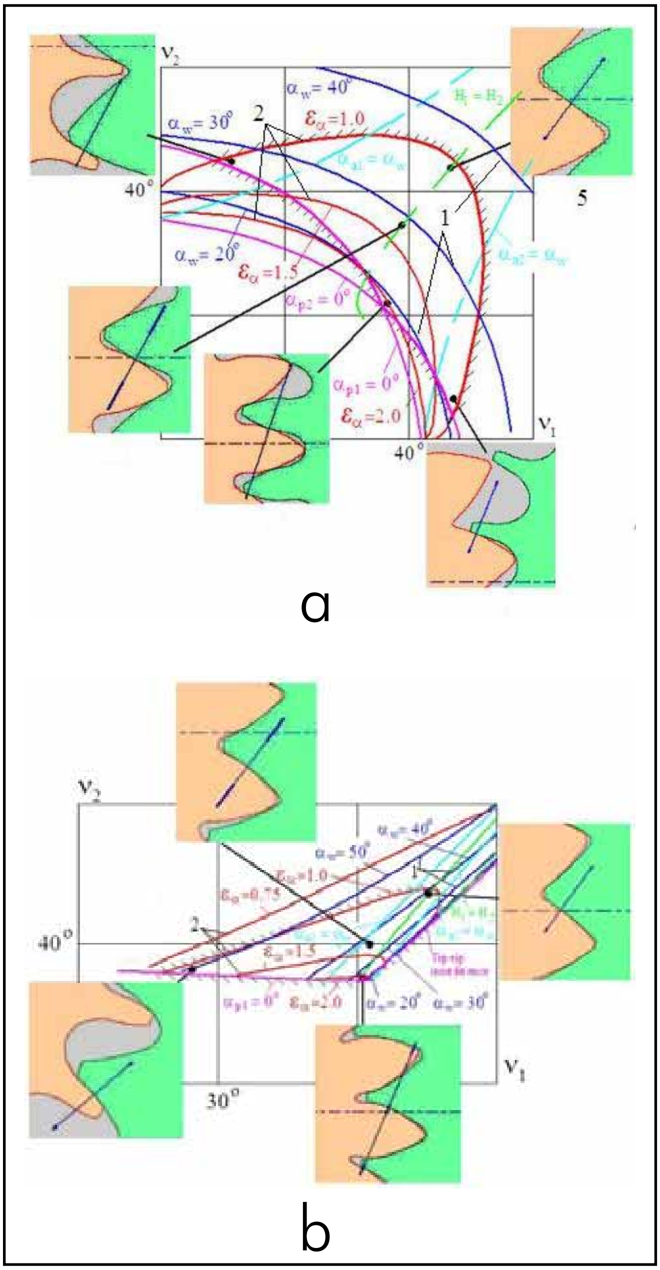

Figure 3—Area of existence for the pinion and gear with n_1 = 18 and n_2 = 25; m_{a1} = 0.25 and m_{a2} = 0.35; Accordingly—*a*: external gearing; *b*: internal gearing; 1: family of the pressure angle isograms α_w = const.; 2: family of the contact ratio isograms ε_α = const.; interference isograms α_{p1} = 0, α_{p2} = 0, and "tip-tip" (for internal gearing); maximum mesh efficiency isograms $H_1 = H_2$; $\alpha_{a1} = \alpha_w$ and $\alpha_{a2} = \alpha_w$: isograms separating the gear meshes with the pitch point laying on the active portion of the contact line.

The pitch point position isograms separate an area of existence into three zones:

- with the pitch point position before the active part of the tooth contact line;
- with the pitch point position on the active part of the tooth contact line (typical for most gears);
- with the pitch point position after the active part of the tooth contact line.

The pitch point position isograms' equations for external gearing are from (Ref. 2 and 4):

isogram $\alpha_{a1} = \alpha_w$,

$$(inv(\alpha_{a2}) - inv(\alpha_{a1})) \times n_2 + m_{a1} + m_{a2} \times \frac{\cos\alpha_{a2}}{\cos\alpha_{a1}} - \pi = 0 \qquad (15)$$

isogram $\alpha_{a2} = \alpha_w$,

$$(inv(\alpha_{a1}) - inv(\alpha_{a2})) \times n_1 + m_{a1} \times \frac{\cos\alpha_{a1}}{\cos\alpha_{a2}} + m_{a2} - \pi = 0 \qquad (16)$$

The pitch point position isogram's equations for internal gearing are from (Refs, 2, 3 and 5):

isogram $\alpha_{a1} = \alpha_w$,

$$(inv(\alpha_{a2}) - inv(\alpha_{a1})) \times n_2 - m_{a1} - m_{a2} \times \frac{\cos\alpha_{a2}}{\cos\alpha_{a1}} + \pi = 0 \qquad (17)$$

isogram $\alpha_{a2} = \alpha_w$ is also defined by equation 16.

The maximum mesh efficiency isogram is defined by condition of the equal specific sliding velocities at the tips of the mating gear teeth $H_1 = H_2$ (Ref. 6). These equations are:
for external gearing:

$$\tan\alpha_{a1} - u \tan\alpha_{a2} + (u-1) \times \tan\alpha_w = 0 \qquad (18)$$

for internal gearing:

$$\tan\alpha_{a1} + u \tan\alpha_{a2} - (1+u) \times \tan\alpha_w = 0 \qquad (19)$$

Area of existence for external gearing (Fig. 3a) is limited by the interference isograms and isogram of the minimum contact ratio (for spur gears it is 1.0). Area of existence of the internal gear pair can also be limited by the "tip-tip" interference isogram.

Every point of the area of existence presents a gear pair with a certain set of the geometric parameters. A few of these gear pairs are shown in Figure 3. Some of them do not look conventional, but they may be practical for some applications.

Area of existence is much greater than the block-contour (Fig. 4) of any particular generating rack. It actually includes any gear pair combinations, generated by all possible block-contours and also the gear pairs, where two different racks generate the mating gears.

Area of Existence for Asymmetric Gearing

The design intent of asymmetric gearing is to improve performance of primary drive profiles at the expense of performance for the opposite coast profiles. The coast profiles are unloaded or lightly loaded during a relatively short work period. Asymmetric tooth profiles also make it possible to simultaneously increase the contact ratio and operating pressure angle beyond conventional gears' limits.

Direct Gear Design represents the asymmetric tooth form by two involutes of two different base circles (Refs. 7 and 8), with the arc distance between them and tooth tip circle describing the gear tooth (Fig. 5). The equally spaced teeth form the gear. The fillet between the teeth is not in contact with the mating gear teeth, but this portion of the tooth profile is critical because it is the area of the maximum bending stress concentration. The fillet profile is designed independently, providing minimum bending stress concentration and sufficient clearance with the mating gear tooth tip in mesh.

The relation between involute profile angles of opposite flanks of an asymmetric tooth is:

$$\frac{\cos\alpha_{xc}}{\cos\alpha_{xd}} = \frac{d_{dc}}{d_{bd}} = k \geq 1.0 \qquad (20)$$

where α_{xd} and α_{xc} are involute profile angles at the $d_x \geq d_b$ diameter. Then:

$$\frac{\cos\alpha_{ac1,2}}{\cos\alpha_{ad1,2}} = \frac{\cos\alpha\nu_{c1,2}}{\cos\alpha\nu_{d1,2}} = \frac{\cos\alpha_{wc}}{\cos\alpha_{wd}} = \frac{d_{dc}}{d_{bd}} = k \geq 1.0 \qquad (21)$$

where k is the asymmetry coefficient.

If $d_{bd} = d_{bc}$, $k = 1.0$ and tooth is symmetric.

The area of existence of asymmetric gears (Fig. 6) is built very similarly to the area of existence of symmetric gears. It basically presents an overlay of two areas of existence: one for the drive flanks and another for the coast flanks of the asymmetric tooth.

The isogram equations for asymmetric gears are very similar to the equations for the symmetric gears.

Application of Area of Existence

A computer program generates the area of existence of involute gears for the given numbers of teeth n1 and n2, relative tooth tip thicknesses m_{a1} and m_{a2}, and asymmetry coefficient k. Then, any selected point in the area presents a

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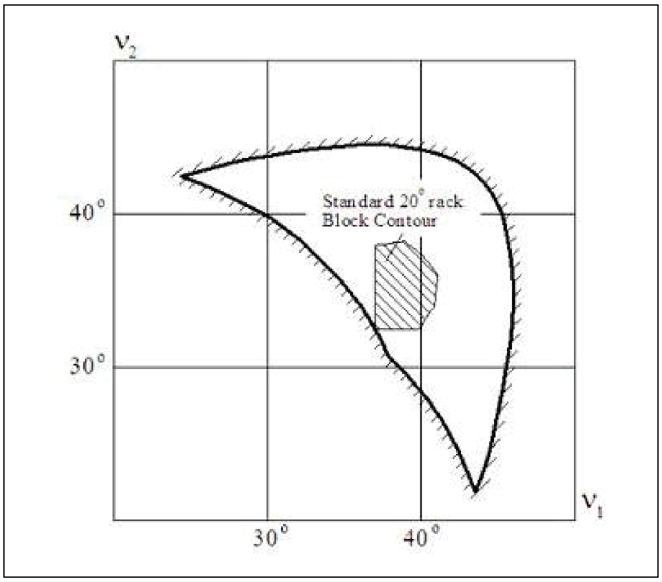

Figure 4—Area of existence for the gear pairs with n_1 = 22 and n_2 = 35 and their standard 20° pressure angle generating rack block-contour.

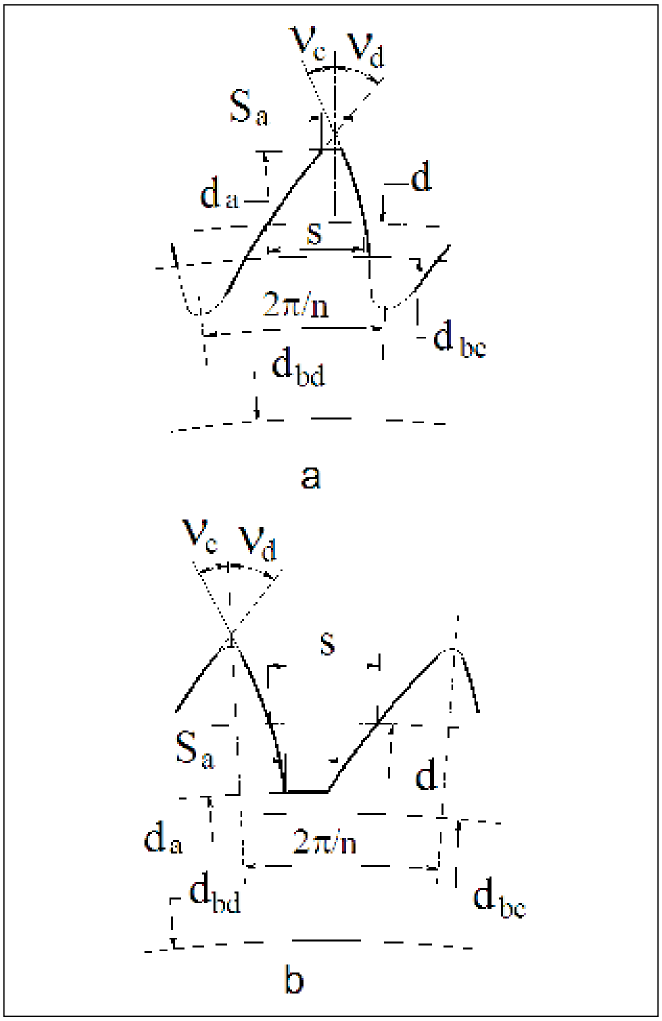

Figure 5—Asymmetric tooth profile (fillet portion red); a: external tooth; b: internal tooth; d_a: tooth tip circle diameter; d_b: base circle diameter; d: reference circle diameter; S: circular tooth thickness at the reference diameter; v: involute intersection profile angle; S_a: circular tooth thickness at the tooth tip diameter; subscripts "d" and "c" are for the drive and coast flanks of the asymmetric tooth.

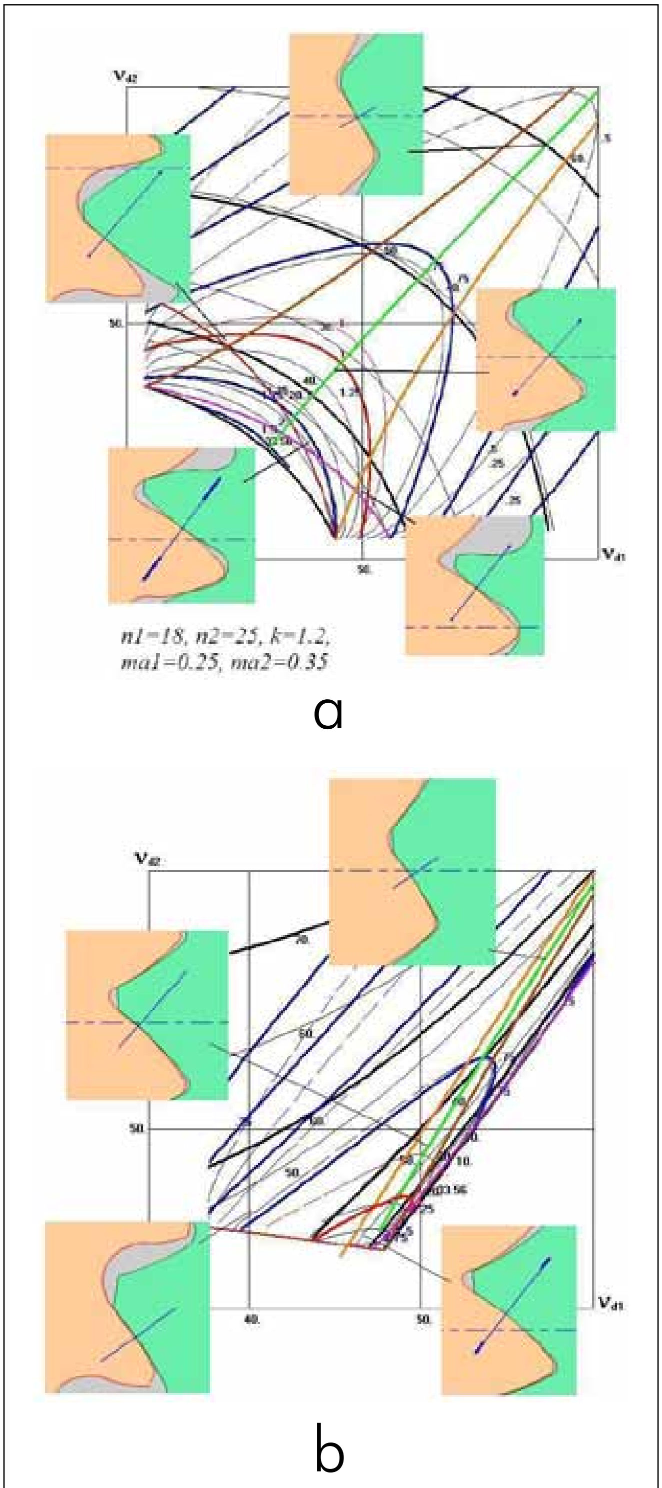

Figure 6—Area of existence for the asymmetric pinion and gear with: n_1= 18 and n_2 = 25; m_{a1} = 0.25 and m_{a2} = 0.35; and k = 1.2. a: for external gearing; b: internal gearing. The isograms related to the drive flank meshes are thick, the isograms related to the coast flank meshes are thin.

set of gear pair mesh parameters, considering its module (or its diametral pitch) and the face widths of the mating gears equal to one. Selection of the relative tooth tip radii and construction of the fillets between the teeth complete the gear geometry definition.

The relative tooth tip radii are: $r_{a1,2} = R_{a1,2}/m$ in the metric system and $r_{a1,2}= R_{a1,2}$ x DP in the English system, where $R_{a1,2}$ are the tooth tip radii of the mating gears. Typically, thicknesses $m_{a1,2}$ are in the range 0.00–0.05.

In traditional gear design, the fillet profile is typically a trajectory of the pre-selected (usually standard) generating gear rack. Any point of the block-contour presents the gear pair with completed (including the fillet) tooth profiles. In Direct Gear Design, the tooth fillet profile is a subject of optimization to minimize bending stress concentration (Refs. 9–10). However, the tooth fillet profile optimization is a time-consuming process that is used for the final stage of gear design. It is not practical for browsing the area of existence, analyzing many sets of gear pairs in limited time period. The tooth fillet profile should be quickly constructed, without tooth tip-fillet interference, and provide relatively low bending stress concentration. In order to achieve this, the virtual ellipsis arc is built into the tooth tip that is tangent to the involute profiles at the tip of the tooth. As a result, the tooth fillet profile is a trajectory of the mating gear tooth tip virtual ellipsis arc (Fig. 7). This fillet profile can be called "pre-optimized" because it provides lower bending stress concentration than the standard rack-generated fillet profile.

The fillet profile construction completes the mating gears teeth geometry definition. This allows the program to demonstrate an animation of the gear mesh right after selection (clicking on) any point of the area of existence.

The next step of area of existence analysis is the calculation of the maximum contact and bending stresses. This stress analysis program procedure requires an input of the operating module (or operating diametral pitch for English system), the face widths for both mating gears and the pinion torque. The modulus of elasticity and Poisson ratio are also required to calculate the Hertzian contact stress. The proprietary 2D FEA sub-routine is used for definition of maximum bending stress for both mating gears.

This program assists in finding a suitable gear solution for a particular application, for example:

1. Heavily loaded low-speed gears: Appropriate gears are at intersection of the maximum pressure angle isogram and the maximum mesh efficiency isogram.

2. Lightly loaded high-speed gears: They can be found at intersection of the high contact ratio (ε_α >2.0) isogram and the maximum mesh efficiency isogram.

3. Dissimilar material gears, like a metal pinion and a plastic gear: In this case, the metal pinion should have the minimum and the plastic gear the maximum relative tooth thickness at the tooth tip diameter. The pressure angle should be relatively low. This allows making the plastic gear tooth thicker and the metal pinion tooth thinner to balance

the bending strength of the mating teeth.

4. Self-locking gears: These parallel axis gears work essentially like worm gears. The solution can be found at a very high pressure angle (α_w >> 60°, gears are helical) and with pitch point position after the active part of the tooth contact line.

Conclusions

The area of existence and its program allow for quickly defining limits of parameter selection of involute gears, locating feasible gear pairs, animating them and reviewing their geometry and stress levels. Benefits of using the area of existence are:

- consideration of all possible gear combinations;
- instant definition of the gear performance limits;
- awareness about non-traditional, "exotic" gear design options;
- quick localization of area suitable for particular application;
- optimization of the gear design solution.

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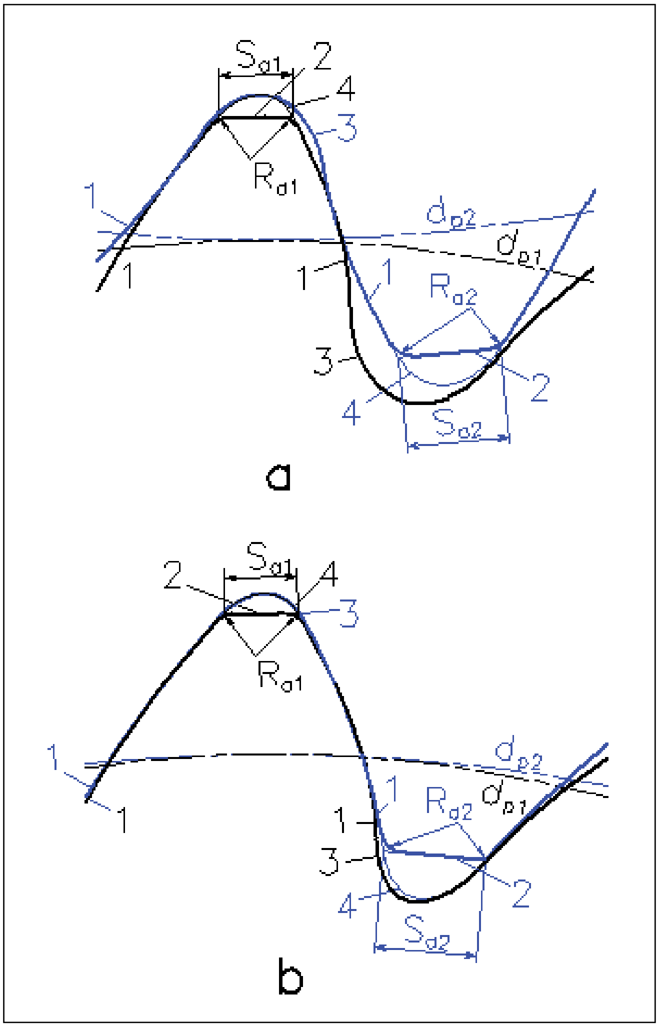
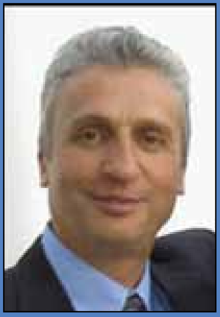


Figure 7—The fillet profile construction. a: external gears; b: internal gearing; 1: involute profiles; 2: tooth tip lands; 3: fillet profiles; 4: ellipsis arcs that are used to generate the fillet profiles.



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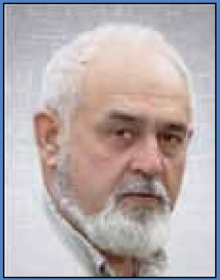

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