The Math of Noncircular Gearing

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Noncircular gearing is not new. There are well-documented articles covering standard and high order elliptical gears, sinusoidal gears, logarithmic spiral gears, and circular gears mounted eccentrically. What these designs have in common is a pitch curve defined by a mathematical function. This article will cover noncircular gearing with free-form pitch curves, which, of course, includes all the aforementioned functions. This article also goes into the generation of teeth on the pitch curve, which is not usually covered in the technical literature. Needless to say, all this is possible only with the help of a computer.

When the machine designer requires a mechanism with cyclic speed or motion variations, he will usually choose between a cam/follower motion, some linkage arrangement, or noncircular gearing. Because of the perceived design and manufacturing difficulties of noncircular gears, they will be the last option considered.

Noncircular gearing provides a greater variety of possible motions than cam arrangements. It allows linear or rotational input and also linear or rotational output, usually with a greater range of motion than a cam. To magnify the eccentricity, there are even compound noncircular gears. They are capable of transmitting power as well as motion, and the wear distribution of noncircular gearing is often better than a cam system for a given lubricity. Manufacturing costs are higher for noncircular gears if they are cut with a hob or

Elliptical and other non circular gears.
Courtesy of Cunningham Industries, Inc.

shaper cutter, but for high-volume stamped, molded, broached, or extruded parts, the costs are comparable to cam systems. Prototypes are often cut by wire EDM.

The design starts the same way you would design a cam, by knowing the required output displacement, velocities and accelerations, and calculating the pitch curves accordingly. The pitch curves may be closed for continuous motion, or open for motion limited to less than one revolution. This article starts with the assumption that you have done the preliminary work and know the pitch curve form. However, you may wish to use the math presented here for trial and error design work.

Noncircular gearing shares many characteristics with both ordinary gearing and cams. The pitch circles of ordinary gears roll together without slipping and are tangent at the line of centers. Similarly, the pitch curves of noncircular gears roll without slipping and are tangent at the line of centers. Noncircular gears may be external or internal. They can also be helical, although this adds greatly to the complexity of their manufacture. The universal law of gear tooth action applies to both ordinary gearing and noncircular gearing. This law states that the common normal through the contact point of two gear teeth in mesh must pass through the pitch point (the pitch point being the contact point of the two pitch curves). Applying this law determines the tooth profile of a mating gear, whatever arbitrary tooth profile may be chosen for the known gear. Also note that if the tooth profile is such that the normal at some point does not intersect its pitch curve, there is no solution for the mating tooth at that point.

Ordinary gears are always associated with a rack form whose pitch line rolls with the gear pitch circle. Similarly, noncircular gears are associated with a rack whose pitch curve rolls with the noncircular gear pitch curve. This rack association is important because two gears which mesh with the opposite sides of a rack profile,
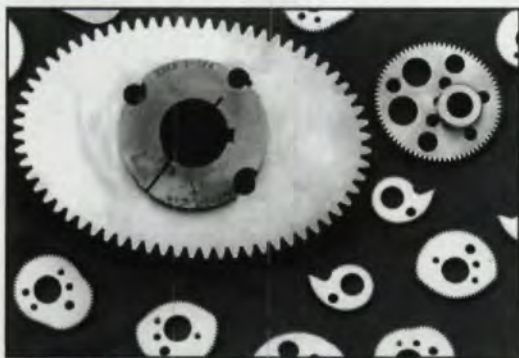
will mesh with each other. This implies that we only have to know the mathematical relation between a gear and a rack instead of having to know the relation between two gears.

Another aspect of the rack is that, if the rack teeth are straight-sided and all have the same pressure angle, the center distance mounting of the system is very forgiving—just like the mounting of ordinary involute gears. Unfortunately, most designs have too much rise to allow this, and the rack teeth pressure angles must vary to follow the pitch curve. Because each noncircular gear tooth is wedded to a particular mating gear or rack space, the teeth do not need to be all alike. There is no restraint on the thickness, height or pitch of the teeth as long as a contact ratio of at least one is maintained. Improved mesh may be achieved if the gear's dedendum can be reduced on parts of the pitch curve that are closest to the center of rotation. This relationship is very much the same as in ordinary gearing where a rule of thumb is that the ratio of the gear dedendum to the gear pitch radius should be less than 0.14 to avoid undercutting at the gear root.

Cams and noncircular gears also share a minimum size restraint in their design. For cams, it is a well-known fact that when the ratio of the total rise to the cam minor radius increases to a critical level, there will be a point where the pressure angle of the cam system becomes unacceptably large. Similarly, for a noncircular gear, when the ratio of the rise of the pitch curve to the pitch curve minimum radius increases to a critical level, there will be a point where the angle of obliquity becomes unacceptably large, 45 degrees being the usual limit. A large angle of obliquity also makes it difficult to design proper-meshing gear teeth.

The free-form definition of the pitch curve is usually made in terms of line-arc geometry, but it could also consist of one or more mathematical functions. To apply the math, the curve must be redefined as a closely spaced set of rectangular coordinates for the rack curve or polar coordinates for the gear curve. Typically the spacing will be about .0004 inches or .0005 radians. The calculations are surprisingly simple.

For our analysis, we will always use the gear center as the origin for both the gear form coordinates and the rack form coordinates. The polar coordinates for the gear pitch curve do not have the standard 'textbook' configuration—angles start at zero on the positive Y-axis and proceed clockwise. Figure 1 shows the axis system with a sample closed pitch curve consisting of three arcs
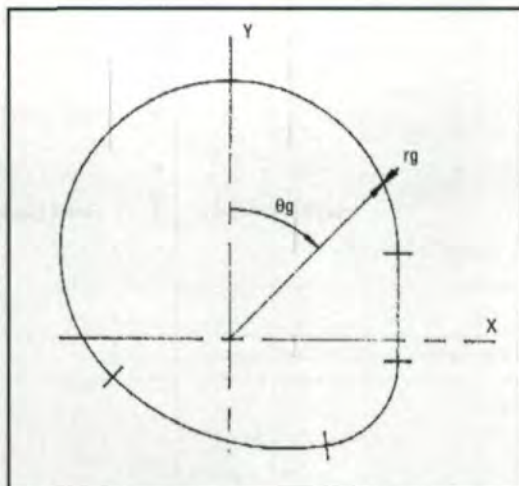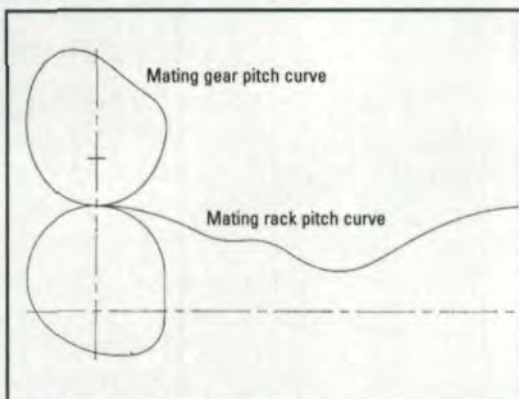

Fig. 1—Axis system and sample pitch curve.

Fig. 2—Mating pitch curves.

and a line. The Glossary (See sidebar) defines the terms and symbols used in this article.

If the given pitch curve is for a gear, then the rack pitch curve is defined by the following algorithm:
1. Select the first point (rg, θg) on the gear pitch curve and set the first rack pitch curve coordinates: xrpl=0, yrpl=rg.
2. If θg equals zero then set XSHF=xrpl.
   Set XSAV=xrpl, YSAV=yrpl, ASAV=θg.

   If there are no more points to process then go to step 3.

   Select the next point (rg, θg) on the gear pitch curve and calculate the next rack pitch curve coordinates:
   xrpl=XSAV + YSAV • sine(θg-ASAV), yrpl=rg.
   Go to step 2.
3. Subtract XSHF from each xrpl coordinate.
   Exit the algorithm.

If the given pitch curve is for a rack, then the gear pitch curve is defined by the following algorithm:
1. Select the first point (xrpl,yrpl) on the rack pitch curve and set the first gear pitch curve coordinates: rg=yrpl , θg=0.
2. If xrpl equals zero then set ROT = θg.

   If there are no more points to process, then go to step 3.

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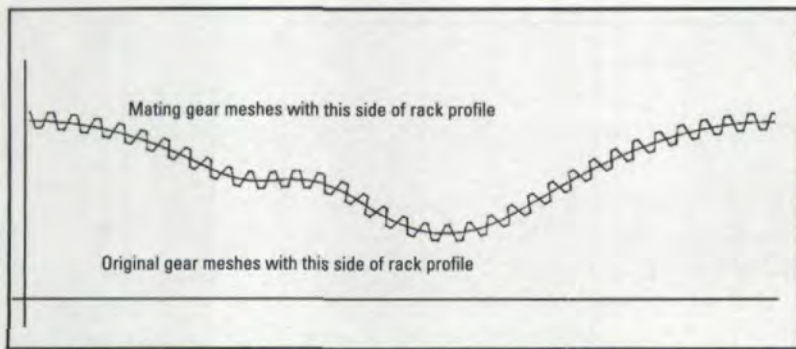
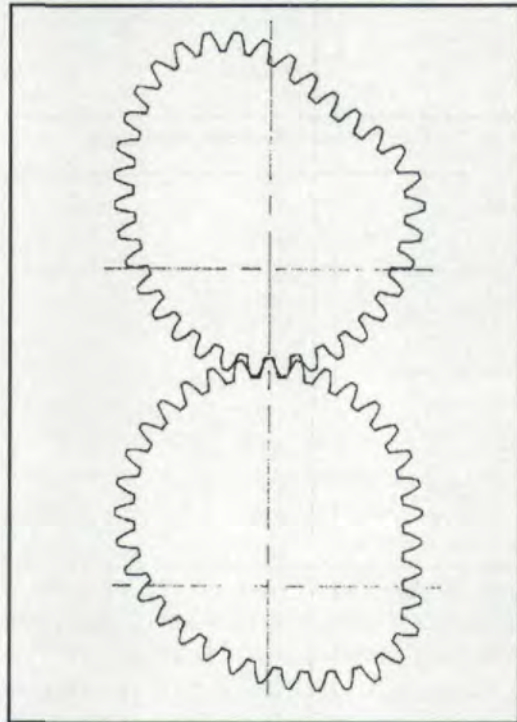*Fig. 3—Rack teeth which will generate teeth for both gears.*

Fig. 4—Meshing noncircular gears.

Set RSAV=rg, ASAV=θg, XSAV=xrpl.

Select the next point (xrpl,yrpl) on the rack pitch curve.

Calculate the next gear pitch curve coordinates:

rg=yrpl, θg=ASAV + arcsin((xrpl-XSAV)/RSAV)
Go to step 2.

3. Subtract ROT from each θg coordinate.
   Exit the algorithm.

Note that to simplify step 2 of both algorithms, we have assumed there will be a xrpl or a θg coordinate of exactly zero. Since this is not ordinarily the case unless you start at zero, you will need to prescan the (xrpl, yrpl) array or the (rg, θg) array and force a coordinate at zero using simple interpolation.

In both cases the resultant pitch curve coordinates should be converted to line-arc geometry using a curve-fitting utility and the length of the pitch curve calculated. If it is not equal to the length of the mating pitch curve within about .0001 inches, then the coordinate point spacing

should be reduced and the algorithm applied again. These small increments simulate the calculus operation that would be applied to a pitch curve defined by a mathematical function. Double precision arithmetic should be used in the computer for these incremental summations in order to avoid loss by rounding.

Figure 2 shows the rack pitch curve developed from the sample in Figure 1 as well as the second mating gear pitch curve developed from the rack pitch curve. In order to maintain the same axis configuration, the rack pitch curve is inverted and shifted vertically to calculate the second mating gear pitch curve. Then the second gear is reinverted to put it in running position with the first gear.

Now we can position teeth along the rack pitch curve. This can be done in several ways. The preferred method is to use a program to scan through the (xrpl, yrpl) array—starting at a measurement that can represent the start of either a tooth or a space, and then continuing to measure through the array, alternately finding tooth and space positions at regular intervals. The measurement accumulates from point to point by calculating the straight-line distance between points, ignoring curvature. This usually requires interpolation between adjacent coordinate points to find an exact tooth position. The result is a table of X, Y coordinates along the rack pitch curve through which the rack teeth must pass. This table is then used in a CAD system to form the straight-sided rack teeth in line-arc geometry, or because of the simplicity of working with straight lines, a program could be written to calculate the rack teeth geometry. This rack tooth geometry should be continuous with root elements and tip elements connecting the flanks. If the corresponding mating gear pitch curve is closed, which signifies a complete revolution, there should be an integral number of teeth whose spacing can be calculated by dividing the length of the pitch curve by the desired number of teeth. The origin for the rack teeth geometry must be at the center of the gear. Figure 3 shows the rack teeth laid out on its pitch curve. The following data can now be used to calculate the mating teeth form:

1. The rack pitch curve form expressed in line arc format.

2. The rack teeth geometry expressed as an array of closely spaced (xrt, yrt) coordinates with slope srt. In practice, you don't need to generate this array at all. Using line-arc format for the rack tooth geometry, pick off one (xrt, yrt, srt) point at a time from the line-arc ele-

ments as the program loop is processed.
3. The xrpl array of rack pitch curve coordinates calculated earlier.
4. The θg array of gear pitch curve coordinates calculated earlier.

The combined (xrpl, θg) array represented by 3 and 4 above is a one-to-one correspondence of rack translation and gear rotation. Use the following algorithm to find the form of the mating teeth expressed as the (xgt, ygt) array:

1. Select a point (xrt, yrt, srt). If there are no more points then exit.
2. Calculate the slope of the normal through (xrt, yrt): sn=-1/srt. This now defines the normal as a line through (xrt, yrt) with slope sn.
3. Calculate the intersection point (xi, yi) where the normal intersects the rack pitch curve. (Use a routine that calculates the intersection point of either two lines or a line and circle. The pitch curve is operated on one segment at a time and checked to see whether the intersection lies within the segment boundaries. If it does, the intersection has been found. The original array of (xrpl,yrpl) points could also be used to find the intersection instead of using its line-arc equivalent.)
4. Scan the xrpl array to find the index, IX, in the array where xrpl equals xi, which will most likely lie between two adjacent values in the xrpl array. Use linear interpolation to find the fractional position: FRACTION = (xi - xrpl(IX)) / (xrpl(IX+1) - xrpl(IX)).
5. Find the corresponding angle in the θg array: ANGLE = θg(IX) + FRACTION * (θg(IX+1) - θg(IX))
6. Calculate the gear tooth coordinates:
xgt=(xrt-xi) • cosine(ANGLE) + yrt • sine(ANGLE)
ygt=yrt • cosine(ANGLE) - (xrt-xi) • sine(ANGLE)
7. Go to step 1

In step 3 of the algorithm, we have simplified the process by assuming that there will be only one intersection point. You may need to consider the possibility of two or more intersections. If there is more than one intersection, only one of them will be correct. You will need to program an inner loop in which separate (xgt, ygt) pairs are calculated for each intersection and then the pair which is closest to the previous point is chosen. The best technique for searching the pitch curve for the intersection point is to start the scan on the pitch curve segment closest to the xrt value under consideration. Then alternately scan the segments left and right of the starting segment.

This gives you the best chance of finding the correct intersection first.

There may be intersection points (xi,yi) missed because the rack pitch curve is not wide enough. If the curve corresponds exactly to a complete revolution of the mating gear, then a portion of the end of the rack pitch curve should be copied, horizontally shifted and appended to the beginning. A portion of the beginning should be copied, horizontally shifted and appended to the end. The shift value applied to the copied portions should be the exact X-extent of the pitch curve. Each portion should be long enough to contain about 3 teeth. The program needs to realize when it has found an intersection in the extended range so that, before searching for a match in the (xrpl, θg) array, xi will have the shift value added or subtracted. Only adjust xi for this search, not for calculations in step 6 of the algorithm.

The gear teeth array (xgt,ygt) is processed through a curve-fitting utility to provide line-arc geometry output and then through a smoothing utility to remove the loops which most often result from tooth generation. Figure 4 shows the two noncircular mating gears, each of which had their teeth generated by the rack teeth of Figure 3.

If you go through the effort of applying the math of this article to your own computer program, a good test of the system is to form a straight, horizontal line rack pitch curve with uniform straight-sided rack teeth. The resultant output should be an ordinary involute gear. ⚙