Efficient Methods for the Synthesis of Compound Planetary Differential Gear Trains for Multiple Speed Ratio Generation

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Abstract:

This article presents an efficient and direct method for the synthesis of compound planetary differential gear trains for the generation of specified multiple speed ratios. It is a train-value method that utilizes the train values of the integrated train components of the systems to form design equations which are solved for the tooth numbers of the gears, the number of mating gear sets and the number of external contacts in the system. Application examples, including vehicle differential transmission units, rear-end differentials with unit and fractional speed ratios, multi-input function generators and robot wrist joints are given.

Introduction

In a simple planetary gear train each planet gear shaft carries one gear; in a compound planetary gear train each planet gear shaft carries two or more gears connected to each other. (See the simple and compound planetary gear trains in Figs. 1 and 3, respectively.) Planet gears in a compound planetary gear train cause speed reduction and change in direction of rotation; planet gears in simple planetary gear trains merely cause change in direction of rotation. Synthesis of compound planetary gear trains can be done by the tabulation method, which utilizes relative motions of gears with respect to the planet arm. In general, it is an iterative process.

Using the equations of motion of planetary gear trains instead of the tabulation method yields a very simple, direct method for analysis as well as for synthesis of both simple and compound planetary gear trains. The equations of motion for a simple planetary gear train, such as shown in Fig. 1, are formed by writing the velocity loop-closure equations for contact points A and C and solving them simultaneously.

Given the following criteria:

$\bar{\omega}_i = \omega_i \bar{k}$ is angular velocity;

r_i is the pitch circle radius of the i^{th} gear or member; member 3 is the planet arm;

$\bar{\omega}3 \equiv \bar{\omega}_a$; $\bar{V}_A = -V_A\bar{i}$; $\bar{V}_c = -V_c\bar{i}$;

$\bar{V}_B = -V_B\bar{i}$; $\bar{V}_{AB} = V_{AB}\bar{i}$; and $\bar{V}_{CB} = -V_{CB}\bar{i}$

where

$V_A = r_2\omega_2$; $V_c = r_5\omega_5$; $V_{CB} = r_4\omega_4 = V_{AB}$ and

$V_B = (r_2 + r_4)\,\omega_a$

The velocity loop-closure equations are

$$\bar{V}_A|_2 = (\bar{V}_B + \bar{V}_{AB})|_4 \qquad (1)$$

$$\bar{V}_C|_5 = (\bar{V}_B + \bar{V}_{CB})|_4 \qquad (2)$$

where the subscripts 2, 4, 5 designate the contact points on members 2, 4, 5, respectively. Equations 1 and 2, in terms of r_i and ω_i, become

$$-r_2\omega_2 = -\omega_a(r_2 + r_4) + r_4\omega_4 \qquad (1a)$$

$$-r_5\omega_5 = -\omega_a(r_2 + r_4) - r_4\omega_4 \qquad (2a)$$

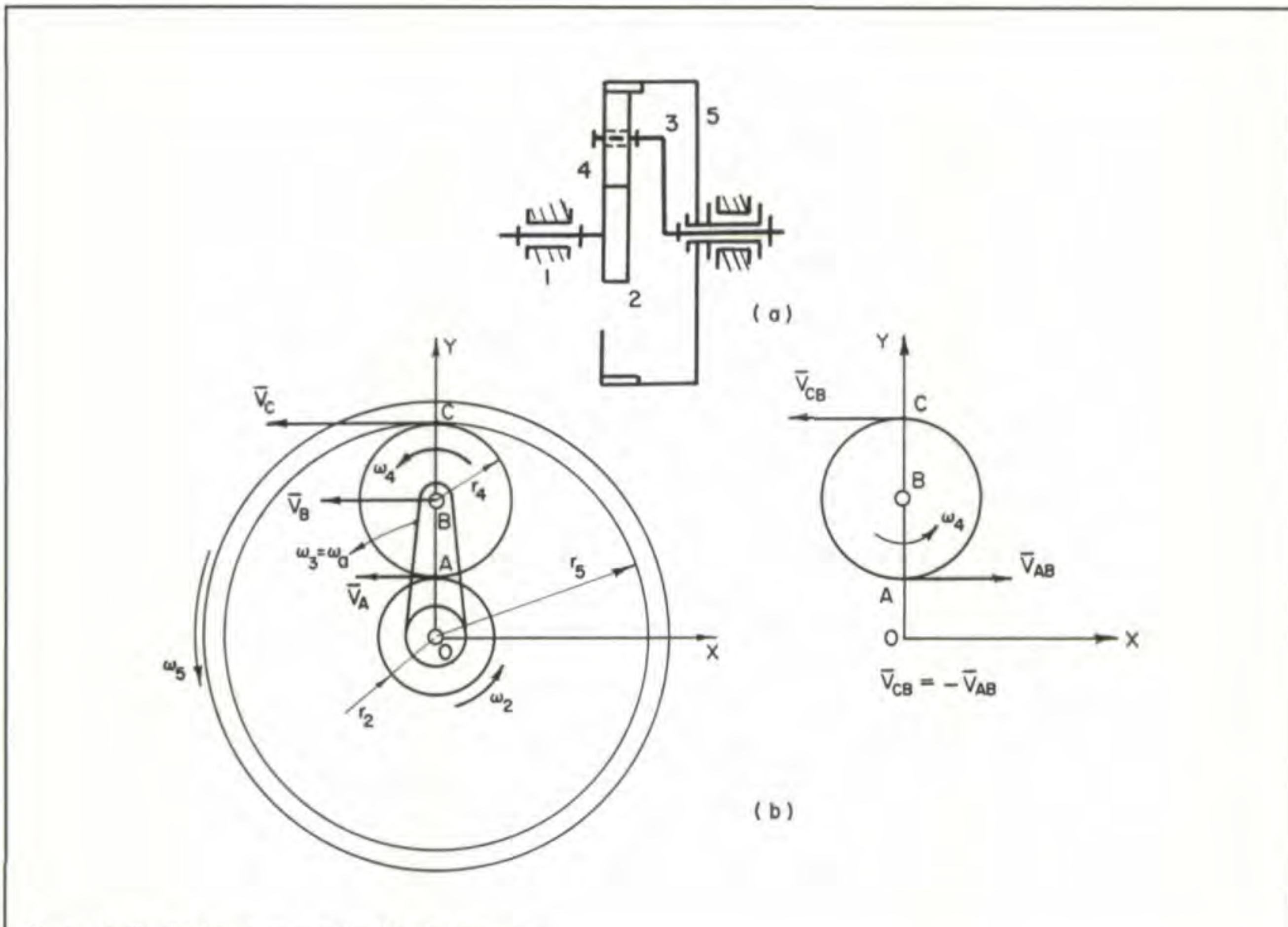

Fig. 1 – (a) Simple planetary gear train, (b) velocity vectors.

Eliminating ω_4 in Equations 1a and 2a we obtain the equations of motion relating speeds of gears 2, 5 and the planet arm as

$$r_2\omega_2 + r_5\omega_5 = 2\omega a\,(r_2+r_4) \tag{3}$$

with the constraint that

$$r_5 = r_2 + 2r_4 \tag{4}$$

Substitute r_5 from (4) into (3) and note that $r_i = N_i/2P$ and $\omega_i = \pi n_i/30$; P is the diametral pitch; n_i is the speed of the i^{th} gear in rpm; n_a is the speed of the planet arm. Then the equation of motion in terms of tooth numbers of gears becomes

$$N_2\,(n_2-n_a) + N_5\,(n_5-n_a) = 0 \tag{5}$$

which is written in the form of train value as

$$e = -\frac{N_2}{N_5} = -\frac{1}{K} = \frac{n_5-n_a}{n_2-n_a} \tag{6}$$

In this equation e is the train value formed as the ratio of the products of tooth numbers of driving gears starting with the input gear, ΣPDVER, to the products of tooth numbers of the driven gears, and ΣPDVEN, when the planet arm is considered stationary. The sign of e, however, is very important in order that no error is introduced during the analysis and synthesis. Thus, e is defined as

$$e = (-1)^q\,\frac{\Sigma PDVER}{\Sigma PDVEN} \tag{7}$$

where q is the number of external contacts of the mating gears. For the gear train in Fig. 1, e is defined as

$$e = -\frac{N_2N_4}{N_4N_5} = -\frac{N_2}{N_5} \tag{8}$$

with q=1 for contact between gears 2 and 4. For the gear train of gears 2 to 7 in Fig. 2, e is defined with q=3 as

$$e = (-1)^3\,\frac{N_2N_4N_5N_6}{N_4N_5N_6N_7} = -\frac{N_2}{N_7} \tag{9}$$

where gear 6 is a double length gear to mate with gears 7 and

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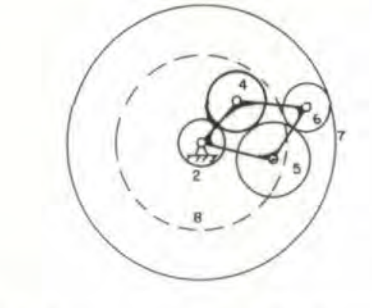


Fig. 2 – Simple planetary gear train with idler planet gears.

8. For the gear train of gears 2 to 8, e is defined with $q=4$,

$$e = \frac{N_2}{N_8} \qquad (10)$$

As observed in Equations 8-10, idler gears do not affect the speed ratio, but their number defines the sign of e.

Subtract Equation 1 from Equation 2 to have

$$\overline{V}_A - \overline{V}_C = 2\,\overline{V}_{AB} \qquad (11)$$

or

$$-r_2\omega_2 + r_5\omega_5 = 2r_4\omega_4 \qquad (12)$$

which, upon substituting r_i and ω_i in terms of N_i and n_i, gives the speed of the planet gear 4 as

$$n_4 = \frac{n_5N_5 - n_2N_2}{2N_4} \qquad (13)$$

Either substituting e and n_5 from Equation 6 and

$$N_5 = 2N_4 + N_2 \qquad (14)$$

from Equation 4 into Equation 14, or considering Equation 1a, we obtain

$$n_4 = n_a - \frac{N_2}{N_4}(n_2 - n_a) \qquad (15)$$

defining n_4 in terms of n_2. Also, either substituting e and n_2 from Equation 6 and N_2 from Equation 14 into Equation 13, *or considering Equation 2a, we obtain*

$$n_4 = n_a + \frac{N_2}{N_5}(n_5 - n_a) \qquad (16)$$

defining n_4 in terms of n_5.

The train value defined by Equation 6 is written in general form as

$$e = \frac{n_L - n_a}{n_f - n_a} \qquad (17)$$

where n_F and n_L are the speeds of the first and last gears in the train considered, and e is written correspondingly. The first gear tooth number in ΣPDVER is the tooth number of the first gear of the train, whichever end of the train one starts with, and the last gear tooth number in the train is the last number in ΣPDVEN.

Similarly, Equations 15 and 16 can also be written in the following general forms: Where n_p is the designated speed of the planet gear, e_{Fp} and e_{Lp} are the train values between the first and planet gear and last and planet gear, respectively.

$$n_p = n_a + e_{Fp}(n_F - n_a) \qquad (18)$$

$$n_p = n_a + e_{Lp}(n_L - n_a) \qquad (19)$$

Note that the coefficient $[(-1)^1 n_2/n_4]$ of $(n_2 - n_a)$ in Equation 15 is the train value e_{24} between gears 2 and 4; the coefficient $[(-1)^0 N_5/N_4]$ of $(n_5 - n_a)$ in Equation 16 is the train value e_{54} between gears 5 and 4.

Equations 18 and 19 form bases for the tabulation method to define the speeds of planet gears. This method simply assumes all moving members are fixed to the planet arm rotating with the arm speed. Then it adds to that speed their individual speeds relative to the planet arm, as if they were ordinary gear trains whose frames were fixed to the planet arm.

As shown in the following section, Equations 17-19 are also the equations of motion for all compound gear trains, bevel gear planetary gear trains and differentials.

Equations of Motion for Compound Planetary Gear Trains

Two compound planetary gear trains are shown in Figs. 3(a) and (b). In (a) the last gear is an internal gear; in (b) it is an external gear. Equations of motion for the compound gear train are written in the same manner as for the simple planetary gear train. Thus, as shown in Fig. 4, velocity loop-closure equations for A and C are

$$\overline{V}_A|2 = (\overline{V}_B + \overline{V}_{AB})|4 \qquad (20)$$

$$\overline{V}_C|6 = (\overline{V}_B + \overline{V}_{CB})|5 \qquad (21)$$

which reduce to

$$-r_2\omega_2 = -\omega_a(r_2 + r_4) + r_4\omega_4 \qquad (22)$$

$$-r_6\omega_6 = -\omega_a(r_2 + r_4) - r_5\omega_5 \qquad (23)$$

Noting that $\omega_5 = \omega_4$ and eliminating them in Equations 22 and 23, we obtain

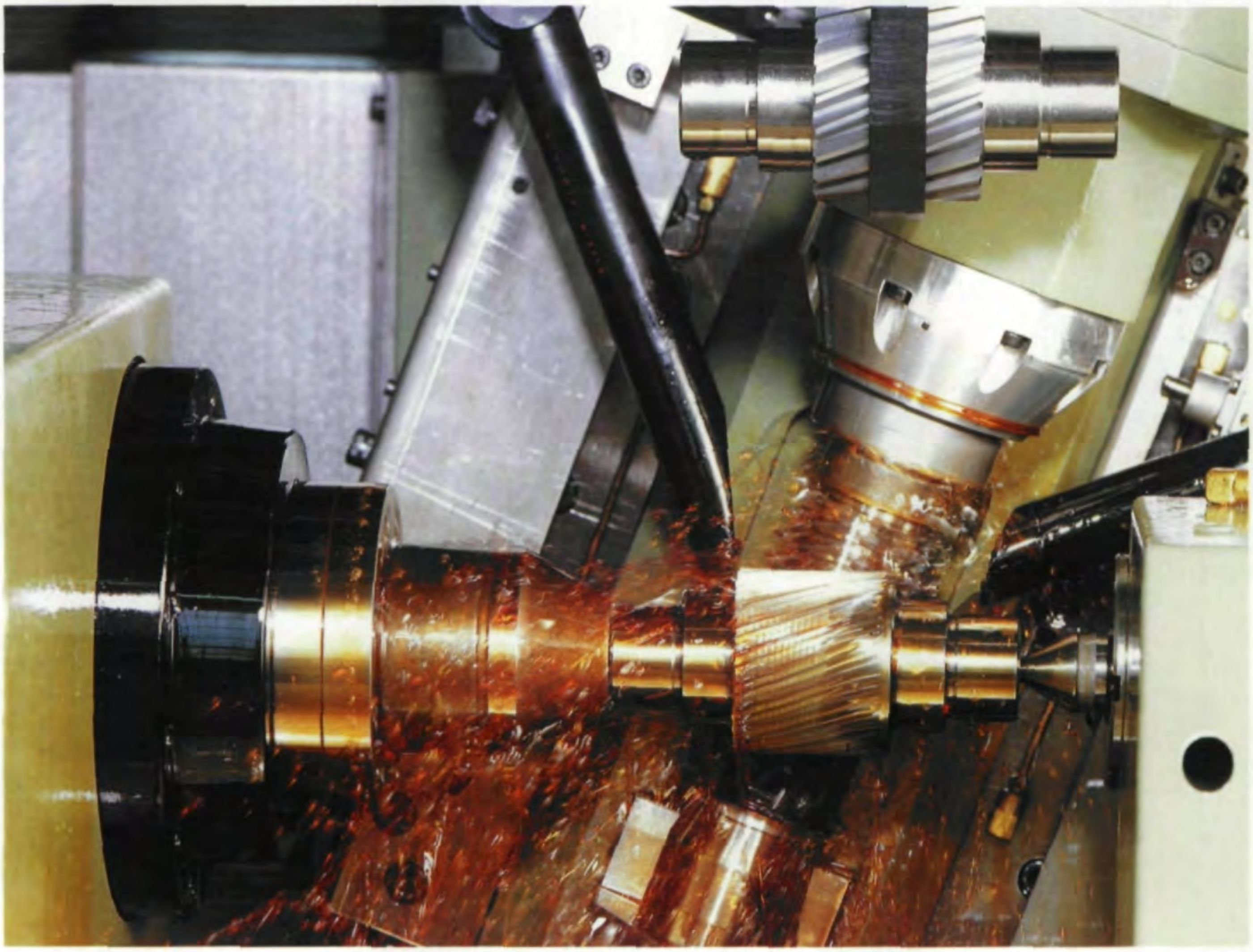
$$r_5r_2(\omega_2 - \omega_a) + r_4r_6\omega_6 = \omega_a[r_4^2 + r_4(r_2 + r_5)] \qquad (24)$$

Noting the following equation of constraint

$$r_2 + r_4 = r_6 - r_5 \qquad (25)$$

or P_{ij} defining diametral pitch between gears i and j,

$$\frac{N_2 + N_4}{2P_{24}} = \frac{N_6 - N_5}{2P_{56}} \qquad (26)$$



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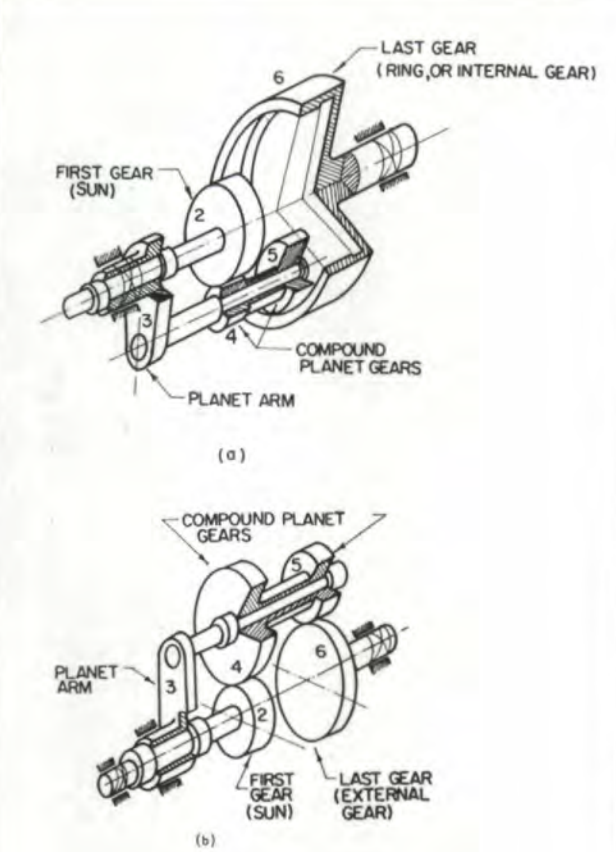

Fig. 3 – Compound planetary gear trains, (a) with internal gear, (b) with all external gears.

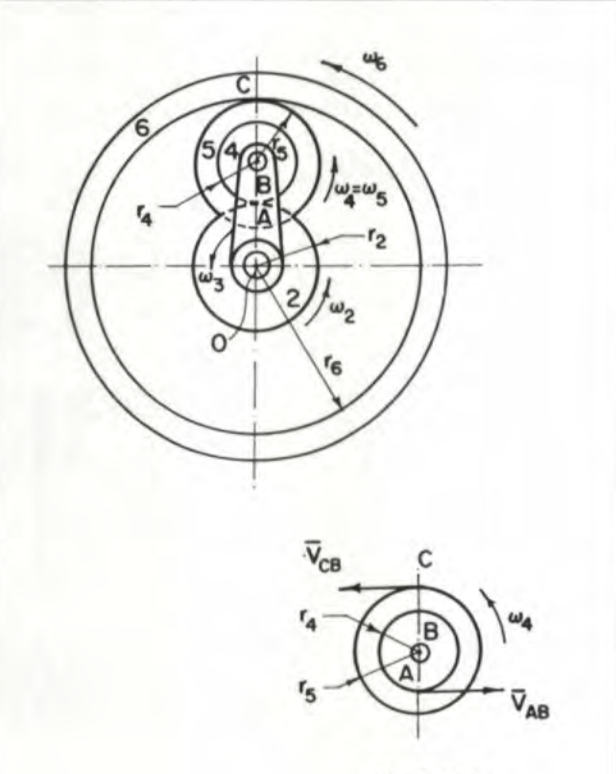

Fig. 4 – Velocities in the compound planetary gear train.

and substituting $r_2+r_5=r_6-r_4$ into Equation 24, we have the equation of motion

$$r_5 r_2 (\omega_2-\omega_a) + r_6 r_4(\omega_6-\omega_a)=0 \tag{27}$$

which, in terms of n_i and N_i, becomes

$$e = -\frac{N_2 N_5}{N_4 N_6} = \frac{n_6 - n_a}{n_2 - n_a} \tag{28}$$

This is in the form of Equation 17 with $q=1$, $n_F=n_2$, $n_L=n_6$, $N_F=N_2$, $N_L=N_6$.

The speed of the planet gear, n_4, is defined in terms of n_2 or n_6 by Equations 22 or 23 as

$$n_4 = n_a - \frac{N_2}{N_4}(n_2-n_a) \tag{29}$$

or

$$n_4 = n_a + \frac{N_6}{N_5}(n_6-n_a) \tag{30}$$

which are also in the forms of Equations 18 and 19, respectively.

Caution: For ordinary gear trains in which the planet arm is fixed, $n_a=0$ and

$$e = \frac{n_L}{n_F} = \frac{1}{S_R} \tag{31}$$

indicating that the train value for ordinary gear trains is the inverse of the speed reduction ratio S_R generated by the gear train. Hence, when a planetary gear train is driven by an ordinary gear train, the ordinary gear train must be analyzed first to determine the input speed to the planetary gear train.

Input-Output Torque Relation

The power equilibrium of a gear train defines the torque multiplication factor (or the mechanical advantage) the planetary gear train generates. Thus,

$$T_{in}\,\omega_{in} + T_{out}\,\omega_{out} = 0 \tag{32}$$

or

$$\frac{T_{out}}{T_{in}} = -\frac{n_{in}}{n_{out}} = -S_R \tag{33}$$

where the subscripts "in" and "out" designate "input" and "output", respectively. Considering the mechanical efficiency of the gear train as η_t, input torque required to generate an output torque is

$$T_{in} = -\frac{T_{out}}{\eta_t S_R} \tag{34}$$

Gear train manufacturers[1] suggest that the mechanical efficiency of two well-lubricated, mating precision gears is about 0.98; for two and three stage speed reductions, it is about 0.97 and 0.96, respectively. Therefore, if there are R sets of gears in a gear train, its mechanical efficiency may be approximated by

$$\eta_t = (0.98)^{[1+0.5(R-1)]} \tag{35}$$

Reduction in efficiency due to energy loss in bearings should also be considered.

Example of Analysis of a Compound Planetary Gear Train

Following is the analysis needed to find the input torques required for each output shaft operation. The gear train shown in Fig. 5 is used to lift 10^6lbf load where the lift ropes wrap around 10" diameter drums. Mechanical efficiency of the system is 0.93. The input shaft of two-lead-worm rotates at 1000 rpm.

Gears 2 and 3 form an ordinary gear train generating the input speed n_4 of the planetary gear train. Hence, $n_4 = n_F$ must be determined first. It is $n_F = 100/3$ rpm.

Since the train value equation (No. 17) contains three shaft speeds, it can only be solved for one speed when the other two speeds are specified. Then, in the planetary gear train of Fig. 5, we must find a sub-planetary gear train receiving two input speeds. Since in this arrangement, gear 9 is fixed, and $n_9 = 0$, the sub-planetary gear train we are looking for is of gears 4, 5, 6, 7, 8 and 9 with $n_L = n_9 = 0$. This train generates the speed of the planet arm, which is one of the output speeds. Thus, for this train we have

$$e_{4-9} = (-1)^2 \frac{N_4 N_6 N_8}{N_5 N_7 N_9} = \frac{n_9 - n_a}{n_4 - n_a}$$

With the tooth numbers shown it reduces to

$$0.00766 = \frac{-n_a}{100/3 - N_a}$$

and

$$n_a = -0.260587 \text{ rpm}$$

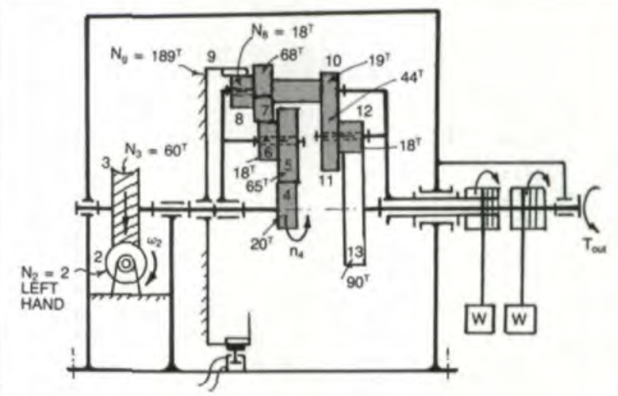

Fig. 5 – Composite planetary gear train analyzed.

The planet arm is rotating in the direction opposite that of gear 4, generating a speed reduction ratio of

$$S_{R_1} = \frac{n_2}{n_a} = -3837.498$$

The second sub-planetary gear train in the system is of gears 4, 5, 6, 7, 10, 11, 12 and 13. Its equation of motion is

$$e_{4-13} = (-1)^4 \frac{N_4N_6N_{10}N_{12}}{N_5N_7N_{11}N_{13}} = \frac{N_{13}-n_a}{N_4-n_a}$$

or

$$0.00703 = \frac{n_{13}+0.260587}{100/3+0.260587}$$

$$n_{13} = -0.024422 \text{ rpm}$$

and

$$S_{R_2} = \frac{n_2}{n_{13}} = -40946.73$$

Speeds of gears 5, 7 and 12 are determined using Equations 18 and 19 as

$$n_5 = n_a - \frac{N_4}{N_5}(n_4-n_a) = -10.5972 \text{ rpm}$$

$$n_7 = n_a + \frac{N_4N_6}{N_5N_7}(n_4-n_a) = 2.4756 \text{ rpm}$$

$$n_{12} = n_a - \frac{N_{13}}{N_{12}}(n_{13}-n_a) = -1.4414 \text{ rpm}$$

Input torque required to lift the load by the planet arm shaft is

$$T_{in_1} = \frac{-5 \times 10^6}{(0.93)(S_{R_1})} = 1401 \text{ in}-\text{lbf}$$

To lift the load by the shaft of gear 13 it is

$$T_{in_2} = \frac{-5 \times 10^6}{(0.93)(S_{R_2})} = 131.3 \text{ in}-\text{lbf}$$

Bevel Gear Planetary Gear Trains

The value of e is determined as shown above. Its sign, however, must be determined by releasing all the gears, retaining the planet arm fixed, and observing the direction in which the last gear rotates relative to the direction of the input rotation. A right hand rule can be followed. Thus, consider the planetary gear train of a robot joint shown in Fig. 6, where for the planetary gear train of gears 2, 4, 5

$$e_{2-5} = -\frac{N_2N_4}{N_4N_5} = \frac{0-n_a}{n_2-n_a} = -\frac{1}{4}$$

and

$$n_a = n_2/5$$

generating speed reduction.

For the planetary gear train of gears 2, 4, 6, 7

$$e = -\frac{N_2N_6}{N_4N_7} = \frac{n_7-n_a}{n_2-n_a} = -2.3935$$

and

$$n_7 = -1.7148n_2$$

generating a speed increase (over drive ratio).

A bevel gear planetary gear train with only sun and planet gears is shown in Fig. 7. The equation of motion for the train is given by Equation 19 as

$$n_4 = n_a - \frac{N_2}{N_4}(n_2-n_a)$$

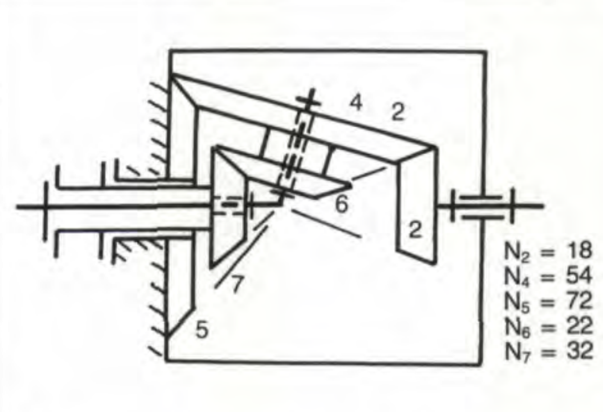

Fig. 6 – Compound bevel gear planetary gear train.

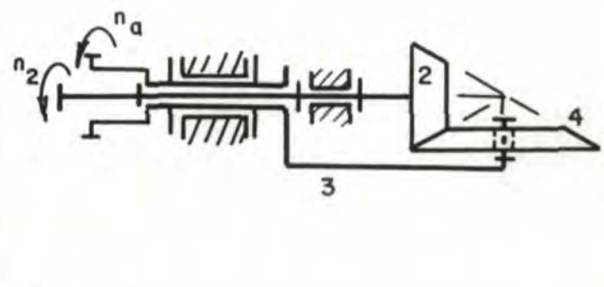

Fig. 7 – Planet gear motion generator.

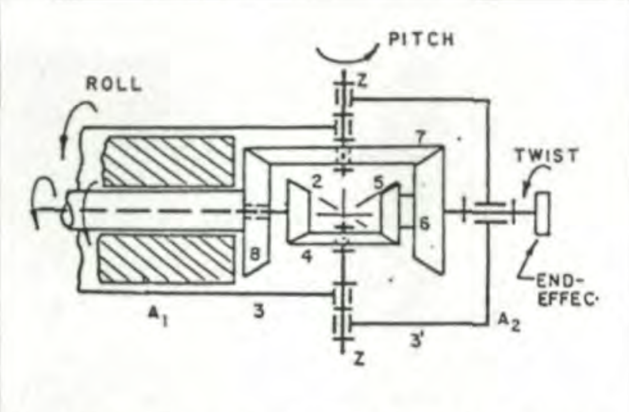

Fig. 8 – Planetary gear train generating roll, pitch and twist motions of a robot wrist joint.

Robot Wrist Joint Using Planetary Gear Trains

Fig. 8 shows another bevel gear planetary gear train used in robot wrist joints. It has three input shafts, shafts of arm A_1, gears 2 and 8[2]. Arm A_1 rotates the joint to reposition the z-z axis about which the arm A_2 rotates when gear 2 or 8 or both rotate. The shaft of gear 5 is the end-effector. With $n_{A_1}=0$,

$$n_4 = -n_2 \frac{N_2}{N_4} \quad (36)$$

and

$$n_7 = n_8 \frac{N_8}{N_7} \quad (37)$$

provide input to the planetary gear train of gears 4, 5, 6, 7 and arm A_2. Its train value is

$$e_{4-7} = -\frac{N_4}{N_5}\frac{N_6}{N_7} = \frac{n_7 - n_{A_2}}{n_4 - n_{A_2}} \quad (38)$$

which defines speed of arm A_2. Substituting n_4 and n_7 from Equations 36 and 37, n_{A_2} is defined in terms of the input speeds as

$$n_{A_2} = \frac{1}{1 + \frac{N_4 N_6}{N_5 N_7}} \left[n_8 \frac{N_8}{N_7} - n_2 \frac{N_2 N_6}{N_4 N_5} \right] \quad (39)$$

The speed of the end-effector is

$$n_5 = n_{A_2} - \frac{N_4}{N_5}(n_4 - n_{A_2}) \quad (40)$$

As observed in Equation 39, in order to retain $n_{A_2} = 0$, n_2 and n_8 must satisfy

$$n_8 = \frac{N_2 N_6 N_7}{N_4 N_5 N_8} n_2 \quad (41)$$

Synthesis of Compound Planetary Gear Trains

As observed in the examples for analysis of planetary gear trains, Equation 17 can be solved to generate only one shaft speed by a sub-planetary gear train. The objective of synthesis is to find the number of mating gears in the train, their tooth numbers and the number of external contacts q to yield the value of e. The three speed reductions a compound gear train can generate with single input speed are

$$S_{R1} = \frac{n_F}{n_a} = \frac{e-1}{e}, \; n_L = 0 \quad (42)$$

$$S_{R2} = \frac{n_F}{n_L} = \frac{1}{e}, \; n_a = 0 \quad (43)$$

$$S_{R3} = \frac{n_L}{n_a} = (1-e), \; n_F = 0 \quad (44)$$

The split input case with two inputs forms the fourth type of speed generation. Inversion generating S_{R2} is a reverted ordinary gear train. S_{R1} and S_{R2} are large speed reductions. S_{R3} is the smallest reduction. Depending on the sign of e, either S_{R1} or S_{R2} is the forward and the largest reduction. For example, if $e<0$, $S_{R1}>0$ and largest, $S_{R2}<0$ and intermediate; if $e>0$, $S_{R2}>0$ and largest, $S_{R1}<0$ and intermediate. S_{R3} is always the smallest forward reduction when $e<1$.

During the synthesis one should keep in mind that the tooth ratio between any two mating spur gears and straight bevel gears should be retained below 8 to assure proper contact, sufficiently large contact ratio and low noise level. Larger tooth ratios may be tolerated for internal and helical gears due to increased contact ratio.

Example: Let us synthesize a compound planetary gear train to generate a speed reduction ratio of +577 from the planet arm shaft with $n_L=0$. By Equation 17

$$e = \frac{n_L - n_a}{n_F - n_a} = \frac{\frac{1}{-577}}{1 - \frac{1}{577}} = -\frac{1}{576}$$

It can be factored into

$$e_1 = -\frac{1}{576} = -\left(\frac{1}{6}\right)\left(\frac{1}{4}\right)\left(\frac{1}{3}\right)\left(\frac{1}{8}\right) = (-1)^3 \frac{N_2}{N_4}\frac{N_5}{N_6}\frac{N_7}{N_8}\frac{N_9}{N_{10}}$$

or

$$e_2 = -\left(\frac{1}{8}\right)\left(\frac{1}{6}\right)\left(\frac{1}{12}\right) = (-1)^3 \frac{N_2}{N_4}\frac{N_5}{N_6}\frac{N_7}{N_8}$$

The last gear in e_1, gear 10, must be an internal gear; in e_2, gear 8 must be an external gear to generate $q=3$. These two

gear trains are shown in Figs. 9 and 10, respectively. In Fig. 9 let

$N_2=20,\ N_4=120,\ N_5=20,\ N_6=80,\ N_7=24,\ N_8=72,\ N_9=24,\ N_{10}=192$

In terms of center distance vectors

$$\overline{C}_{25} = (r_2 + r_4)\,\overline{U}_{25}$$

$$\overline{C}_{57} = (r_5 + r_6)\,\overline{U}_{57},\ \overline{C}_{79} = (r_7 + r_8)\,\overline{U}_{79},$$

and $\overline{C}_{29} = (r_{10} - r_9)\,\overline{U}_{29}$, where $\overline{U}_{ij}$ is the unit vector.

$$|\overline{C}_{29}| \leq |\overline{C}_{25} + \overline{C}_{57} + \overline{C}_{79}| \geq \overline{C}_{29} \qquad \text{(a)}$$

must be satisfied when mounting the planet gears on the planet arm. (See Fig. 9b.) Considering the same diametral pitch for all the gears, Equation (a) demands that, depending on the locations of shafts of gears 5 and 7,

$$N_2 + N_4 + N_5 + N_6 + N_7 + N_8 \geq N_{10} - N_9$$
$$N_2 + N_4 + N_5 + N_6 - N_7 - N_8 \leq N_{10} - N_9$$

and

$$N_2 + N_4 - N_5 - N_6 + N_7 + N_8 \leq N_{10} - N_9$$

must be satisfied.

In Fig. 10 let

$N_2=20,\ N_4=160,\ N_5=20,\ N_6=120,\ N_7=18,\ N_8=216$

which must satisfy

$$|\overline{C}_{27}| \leq |\overline{C}_{25} + \overline{C}_{57}| \geq |\overline{C}_{27}| \qquad \text{(b)}$$

which demands that

$$N_2 + N_4 - N_5 - N_6 \leq N_7 + N_8$$

and

$$N_2 + N_4 + N_5 + N_6 \geq N_7 + N_8$$

must be satisfied.

To generate this speed reduction an ordinary planetary gear train will require $N_L/N_F=278$. With $N_F=20$, $P=6$, one notices the unbelievable gear size: $N_L = 5520$, $d_L = 920''$. This defect with ordinary planetary gear trains is minimized by forming series-connected ordinary planetary gear trains.

The planet arm and the planet gears must be balanced, which is commonly achieved by mounting symmetrical planet gears.

Multiple Speed Reduction Generation

Synthesis of a compound planetary gear train is performed in two forms:

Form 1. All the desired speed reductions are generated simultaneously, without requiring shifting for gear arrangements within the gear train.

Form 2. Only one desired speed reduction is generated at a time, requiring shifting of gears, releasing and activating clutches.

The second form of synthesis yields costly systems, since it requires complex mechanisms, connections and clutches to shift the gears and change the shafts. In general it requires a series connection in which the output of one unit is used to drive the next unit.

Example for Form 1 Synthesis

Let us synthesize a compound planetary gear train to generate the speed reduction ratios, 10, −4 and 2.5 simultaneously.

The largest reduction is generated by S_{R1} or S_{R2}. In a compound planetary gear train it is a preferred trade to retain the

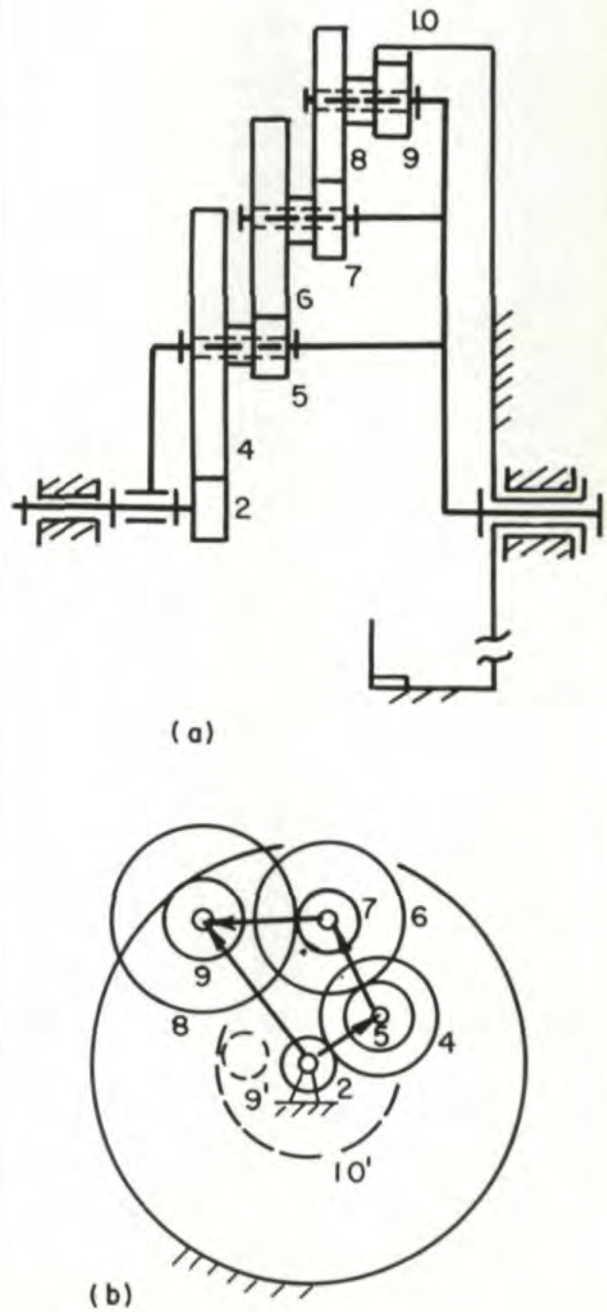

Fig. 9 – Compound planetary gear train generating speed reduction ratio of 577 with eight gears.

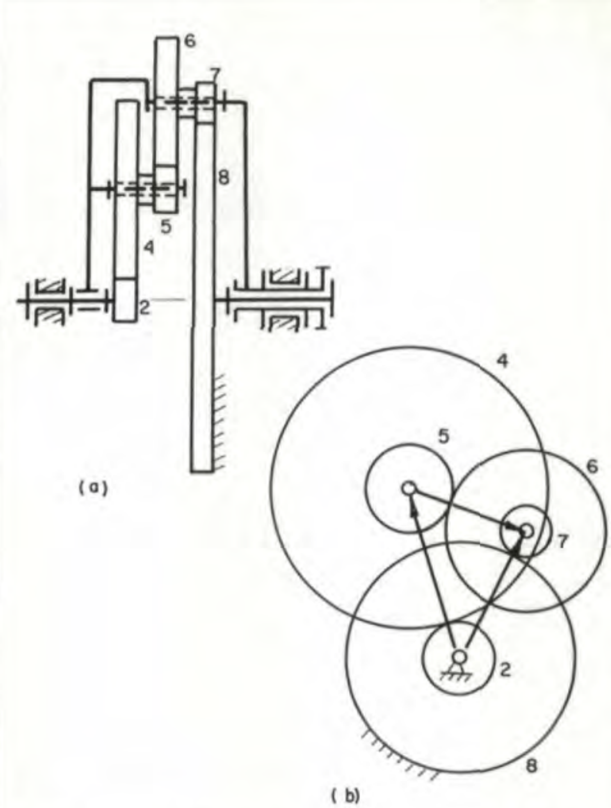

Fig. 10 – Compound planetary gear train generating speed reduction ratio of 577 with six gears.

planet arm active. So let S_{R1} in Equation 42 generate speed reduction 10. Hence,

$$10 = \frac{e_1 - 1}{e_1} \qquad \text{(a)}$$

and

$$e_1 = -\frac{1}{9}$$

or

$$e_1 = \frac{-\frac{1}{10}}{1 - \frac{1}{10}} = -\frac{1}{9} \qquad \text{(b)}$$

Using four gears, the first two having external contact, we form

$$e_1 = \left(-\frac{1}{3}\right)\left(\frac{1}{3}\right) = \left(-\frac{N_2}{N_4}\right)\left(\frac{N_5}{N_6}\right) \qquad \text{(c)}$$

which states that gear 6 is an internal gear. This portion of the gear train is shown is Fig. 11(a). Considering $N'_2=1$, $N'_4=3$, $N'_5=1$, $N'_6=3$, the constraint equation

$$a\,(N'_2 + N'_4) = b\,(N'_6 - N'_5) \qquad \text{(d)}$$

must be satisfied, where

$$a = \frac{1}{2P_{24}}, \; b = \frac{1}{2P_{56}}.$$

To use the same diametral pitch

$$a\,(1+3) = b(3-1) \qquad \text{(e)}$$
$$2a = b$$

Letting $a=1$, $b=2$, we have

$$1 + 3 = 6 - 2 \qquad \text{(f)}$$

defining $N'_2=1$, $N'_4=3$, $N'_5=2$, and $N'_6=6$. Now, both sides of (f) are multiplied by the same number to define the actual tooth numbers. Multiply by 40 (or 20, 24, 30, etc.) to have $N_2=40$, $N_4=120$, $N_5=80$, $N_6=240$.

Using the same planet arm, the other two speed reductions are generated. Thus, by Equation 18, and noting that

$$-\frac{N_2}{N_4} = -\frac{1}{3}$$

must also exist in the sub-gear trains, speed reduction ratio (-4) is generated by

$$e_2 = \left(-\frac{N_2}{N_4}\right) e_B = \frac{-\frac{1}{4} - \frac{1}{10}}{1 - \frac{1}{10}} = -\frac{14}{36} = \left(-\frac{1}{3}\right)\left(\frac{14}{12}\right) \qquad \text{(g)}$$

where

$$e_B = \frac{14}{12}$$

Its positive sign requires another internal gear, gear 8, or two sets of external gears. Choosing the latter,

$$e_B = \frac{14}{12} = \left(\frac{2}{2}\right)\left(\frac{7}{6}\right) = \left(\frac{N_7}{N_8}\right)\left(\frac{N_9}{N_{10}}\right) \qquad \text{(h)}$$

Tooth numbers in (h) must satisfy

$$N_2+N_4 \le N_7 + N_8 + N_9 + N_{10}$$

and

$$N_2+N_4+N_7+N_8 \ge N_9+N_{10}$$

Hence,

$$160 \le C\,(2 + 2 + 7 + 6) \le 240$$

and

$$160 \ge C(7+6-2-2)$$

and

$$9 \le C \le 17$$

letting $C=10$, $N_7 = 20$, $N_8 = 20$, $N_9 = 70$ and $N_{10} = 60$.

This sub-train expands the gear train to the form shown in Fig. 11(b), gears 7 and 8 being the same size.

Speed reduction ratio (+2.5) is generated by

$$\left(-\frac{N_2}{N_4}\right) e_c = \frac{\frac{1}{2.5} - \frac{1}{10}}{1 - \frac{1}{10}} = \frac{3}{9} = \left(-\frac{1}{3}\right)\left(-\frac{1}{1}\right)$$

where

$$e_c = -\frac{1}{1} = -\frac{N_{11}}{N_{12}}$$

gear 12 being an external gear. Since

$$N_2 + N_4 = 160 = N_{11} + N_{12} = d(1+1)$$

$$d = 80, \text{ and}$$

$$N_{11} = N_{12} = 80$$

The final form of the gear train is shown in Fig. 11(c). The gear train may drive three units with the three speeds it generates, or only one unit may be driven by simply using an external clutch coupling or shaft splines shifted axially as shown in Fig. 11(c).

It should be noted that the second speed reduction (−4) could also be generated, forming a gear train that uses $n_7 = n_2$ and the planet speed (1/10) of the first reduction as the split inputs, where the planet arm drives the last gear.

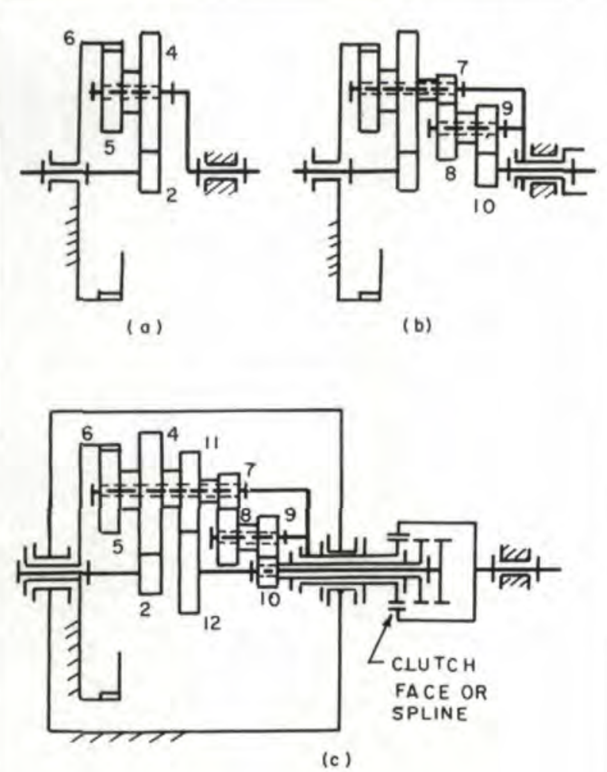

Fig. 11 – Compound planetary gear train synthesized to generate speed reduction ratios 10, −4 and 2.5, simultaneously by form 1 synthesis.

Thus,

$$e_2 = \frac{\frac{1}{10} + \frac{1}{4}}{1 + \frac{1}{4}} = \frac{7}{25} = \left(-\frac{2}{5}\right)\left(-\frac{7}{10}\right) = \left(-\frac{34}{85}\right) -\frac{49}{70}$$

$$= \left(-\frac{N_7}{N_8}\right)\left(-\frac{N_9}{N_{10}}\right)$$

yielding a second compound planetary gear train with $N_7 = 34$, $N_8 = 85$, $N_9 = 49$ and $N_{10} = 70$, which satisfy $(r_7 + r_8) = (r_9 + r_{10})$ with $P_{79} = P_{9,10}$. In this case the planet arm of the first unit must drive gear 10, requiring a more complex shaft arrangement.

Speed reduction 2.5 may be generated similary using split inputs $n_{11} = n_2$ and $n_L = 1/10$. Thus,

$$e_3 = \frac{\frac{1}{10} - \frac{4}{10}}{1 - \frac{4}{10}} = -\frac{1}{2} = -\frac{40}{80} = -\frac{N_{11}}{N_{13}}$$

yielding a simple planetary gear train as the third unit with $N_{11} = 40$, $N_{12} = 20$, $N_{13} = N_{13} = 80$. The new solution has three planetary gear train units with three planet arms and more complex shaft connections. It may be a much larger and more costly large drive system compared to the solution in Fig. 11c.

As noted, Form 1 synthesis is a *stepwise synthesis* for simultaneous speed reduction generation.

Example for Form 2 Synthesis

In this case three gear trains in series will be formed with different shaft connections for each speed reduction generated. Since there are three speeds to be generated, three products of speed ratios are formed using the forms of speed reductions in Equations 42 – 44. One can form 729 combinations of three products of S_{R1}, S_{R2} and S_{R3}, considering each unit causing speed reduction. Noting that the inverse of a speed reduction is an overdrive condition, with the possibility of two overdrive units out of three units, 243 other combinations are possible.

Let us generate speed ratios 10, −4 and 2. Choose the following products:

$$\left(\frac{e_1 - 1}{e_1}\right)\left(\frac{e_2 - 1}{e_2}\right)\left(\frac{1}{e_3}\right) = 10 \quad \text{(a)}$$

$$\left(\frac{e_1 - 1}{e_1}\right)\left(\frac{1}{e_2}\right)\left(\frac{1}{e_3}\right) = -4 \quad \text{(b)}$$

$$\frac{(e_1 - 1)}{e_1}\frac{(1)}{e_2}(1 - e_3) = 2 \quad \text{(c)}$$

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where the last gear of unit 1 is fixed, its planet arm drives the first gear of unit 2, whose planet arm drives the first gear of unit 3, whose arm is fixed, and its last gear generates $S_{R1}=10$ in (a). Unit 1 does not change its fixed shaft condition. In (b) and (c) the arm shaft of unit 1 drives the sun gear of unit 2, whose last gear drives the first gear of unit 3 in (b); it drives the last gear of unit 3 in (c), the first gear being fixed. Several clutches and shaft coupling units are needed to form the shaft arrangements in this case. Solving (a), (b) and (c) simultaneously one finds

$$e_2 = -1.5$$

and

$$e_3^2 - e_3 - 0.5 = 0$$

or the two values of e_3 as

$$e_{31} = 1.3661$$

$$e_{32} = -0.3661$$

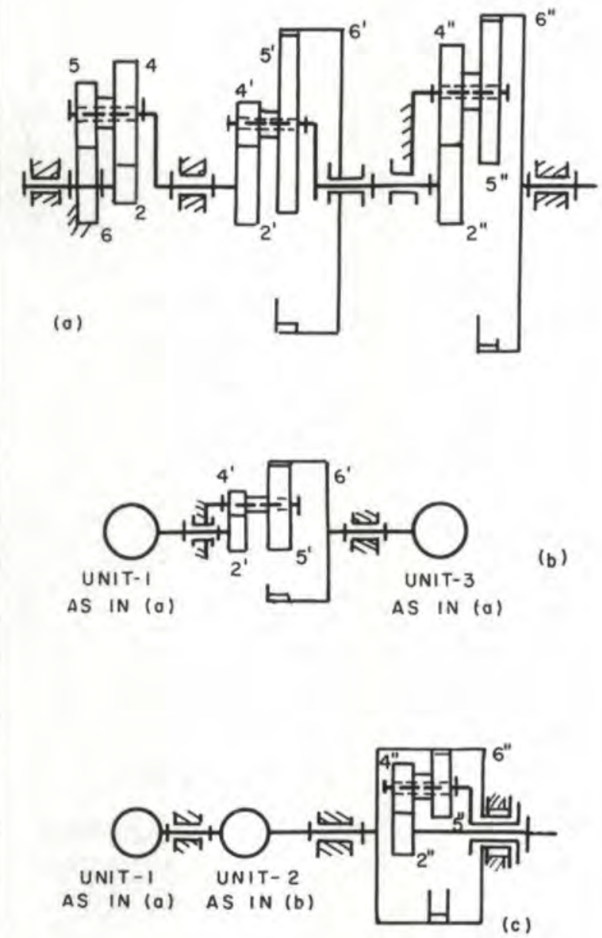

Fig. 12 – Compound planetary gear train synthesized to generate speed reduction ratios 10, −4 and 2 one at a time by form 2 synthesis.

Then,

$$e_{1i} = \frac{1}{1+4\,e_2 e_{3i}}\,,\ i=1,2$$

leading to two solution gear trains with

$$e_{l_1} = -0.1390,\ \text{and}\ e_{l_2} = 0.3129$$

The number of gears, their tooth numbers and the value of q for each solution unit is determined independently of the others. Thus, for unit 2

$$e_2 = -\frac{N'_2\,N'_5}{N'_4\,N'_6} = -1.5 = \left(\frac{-3}{1}\right)\left(\frac{1.5}{3}\right)$$

and N'_6 must be an internal gear. Satisfying

$N'_2+N'_4 = N'_6-N'_5$, $a\,(3+1)=b(3-1.5)$, $a=3$, $b=8$; $N'_2= 54$, $N'_4 = 18$, $N'_5 = 72$, $N'_6 = 144$ for unit 2. Values e_1 and e_3 are satisfied approximately.

Consider the second solution. For unit 1

$$e_{l_2} = 0.3129 = \frac{1}{3.1959} \simeq \left(-\frac{1}{2}\right)\left(\frac{1}{1.6}\right) = \frac{1}{3.2} =$$

$$\left(\frac{-N_2}{N_4}\right)\left(-\frac{N_5}{N_6}\right)$$

in which gear 6 is an external gear, satisfying

$$a(N_2+N_4) = b(N_5+N_6)$$

$$N_2=26,\ N_4=52,\ N_5=30,\ N_6=48.$$

For unit 3

$$e_{3_2} = -\ 0.3661 \cong -\frac{1}{2.7315} \cong \left(\frac{-1}{1.3}\right)\left(\frac{1}{2.1}\right) =$$

$$\left(-\frac{N''_2}{N''_4}\right)\left(\frac{N''_5}{N''_6}\right)$$

in which gear 6 is an internal gear, and satisfying

$$a(N''_2+N''_4) = b(N''_6-N''_5)$$

$$N''_2=110,\ N''_4=143,\ N''_5=230,\ N''_6=483.$$

The three gear trains with their shaft connections to generate the three speed ratios are shown in Figs 12(a), (b) and (c). With the approximated e_3 and e_1, the system generates:

$$10.01,\ -4.004,\ \text{and}\ 2.039.$$

Observing the sizes of gears in unit 3, and noting the existence of three planet arms and several clutch and coupling units and shifting or actuating mechanisms within the system in form 2 synthesis, synthesis as in form 1 is more economical and advantageous. Other products of e_{32} should be searched. A smaller drive with a larger speed error may be formed. For example, try

$$e_{32} \cong \left(-\frac{1}{1.7}\right)\left(\frac{1}{1.6}\right) = -0.36765 = \left(-\frac{10}{17}\right)\left(\frac{10}{16}\right)$$

$$= \left(-\frac{N''_2}{N''_4}\right)\left(\frac{N''_5}{N''_6}\right)$$

and $N''_2 = 30$, $N''_4 = 51$, $N''_5 = 135$, $N''_6 = 216$, generating the speed reduction ratios as 9.9547, −3.98188, 2.00216.

Readers should form solution gear trains for e_{31} and e_{11}.

Differential Gear Trains

When a planetary gear train is driven with two inputs to generate the third speed (split input drive), it is in general called a "differential gear train". Some differential gear trains were already synthesized in the foregoing compound planetary gear train examples. Here, we will see how a differential with two inputs can be synthesized to generate the sum of two variables as a mechanical computer. Rewrite Equation 17 for the planet arm as

$$n_a = A\,n_L - B\,n_F \tag{45}$$

where

$$A = \frac{1}{1-e},\quad B = \frac{e}{1-e}$$

A planetary gear train can be synthesized to generate a function of the form

$$z = ax + by \tag{46}$$

by the planet arm, x and y being the input functions $n_L \equiv x$, $n_F \equiv y$ and $n_a = z$. Consider the following example.

Example: Let us generate

$$z = 2x - 4y \tag{47}$$

Comparing Equation 45 and 47 we have

$$A = \frac{1}{1-e} = 2\,,\ B = \frac{e}{1-e} = 4$$

Using A or B, find e being sure that the other coefficient, B or A, is larger than its required value so that an overdrive unit is not formed. In this case, we use B to find e as

$$e = \frac{4}{5}$$

which yields

$$A = 5$$

Therefore, the input to the last gear of the train must be supplied as

$$x/m$$

where

$$m = \frac{A_{formed}}{A_{desired}} = \frac{5}{2} = 2.5 \tag{48}$$

e = 4/5 is generated by four external gears to supply y. Thus,

$$e = \frac{4}{5} = \left(-\frac{N_2}{N_4}\right)\left(-\frac{N_5}{N_6}\right) = \left(-\frac{36}{36}\right)\left(-\frac{32}{40}\right)$$

and $N_2 = 36$, $N_4 = 36$, $N_5 = 32$, $N_6 = 40$. The desired differential gear train is shown in Fig. 13(a), where $N_7 = 20$, $N_8 = 30$, $N_9 = 50$.

Equation 17 may also be written to generate z by the shaft of the last gear, supplying x on the arm shaft, y on the first gear shaft, as

$$n_L = Dn_a + En_2 \tag{49}$$

with $D = (1-e)$, $E = e$. In that case, to generate the function

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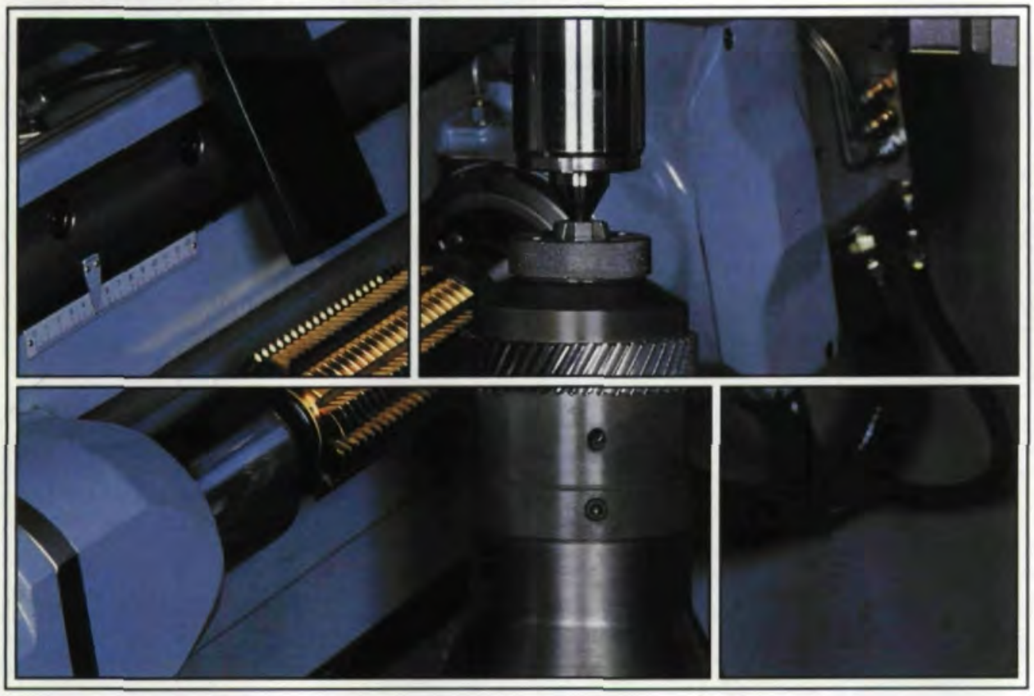

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in Equation 47, $e = -4$, $n_a \equiv x/2.5$ and $n_F = y$. A bevel gear train generating z for this case is shown in Fig. 13 (b), where

$$e = -4 = -\frac{N_2}{N_4}\frac{N_5}{N_6} = -\left(\frac{40}{80}\right)\left(\frac{20}{40}\right)$$

The reader should form another gear train using $e = -1$ supplying $n_a = x$, $n_F = 4y$ as overdrive.

Vehicle Rear End Differentials

The objective in designing a vehicle rear end differential planetary gear train is to generate $e = -1$, so that when the vehicle is making a turn without the planet arm rotating, the outer wheel goes one unit rotation forward as the inner wheel goes one unit rotation backward. Fig. 14 shows a commonly used bevel gear rear end differential, where gear 7 is driven by the universal shaft to supply the input by the planet arm. $N_2 = N_5$; N_4 is of any suitable number and $e = -1$. When gear 2 is held stationary, $n_2 = 0$, and gear 5 is lifted for balancing, one observes

$$-1 = \frac{n_5 - n_a}{-n_a}, \quad n_5 = 2n_a$$

Vehicle rear-end differentials can also be formed using spur gears, provided $e = -1$ is maintained. Figures 15(a) and (b) show two such rear-end differentials, where planet gears

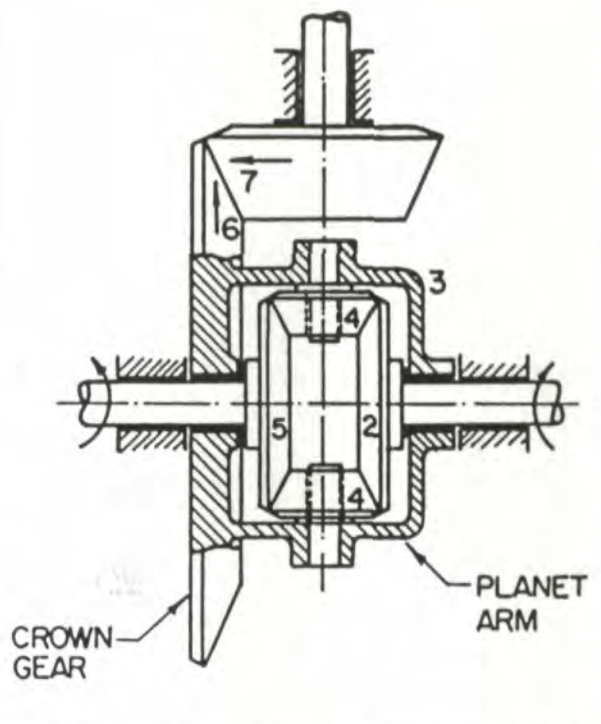

Fig. 14 – Bevel gear planetary gear train differential for vehicle rear-end with $e = -1$.

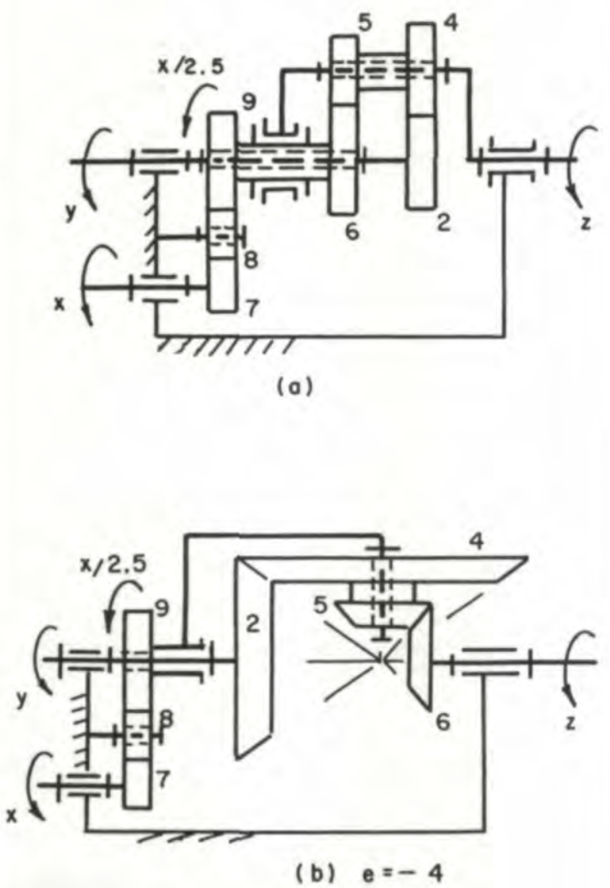

Fig. 13 – $z = ax + by$ function generating planetary gear differentials.

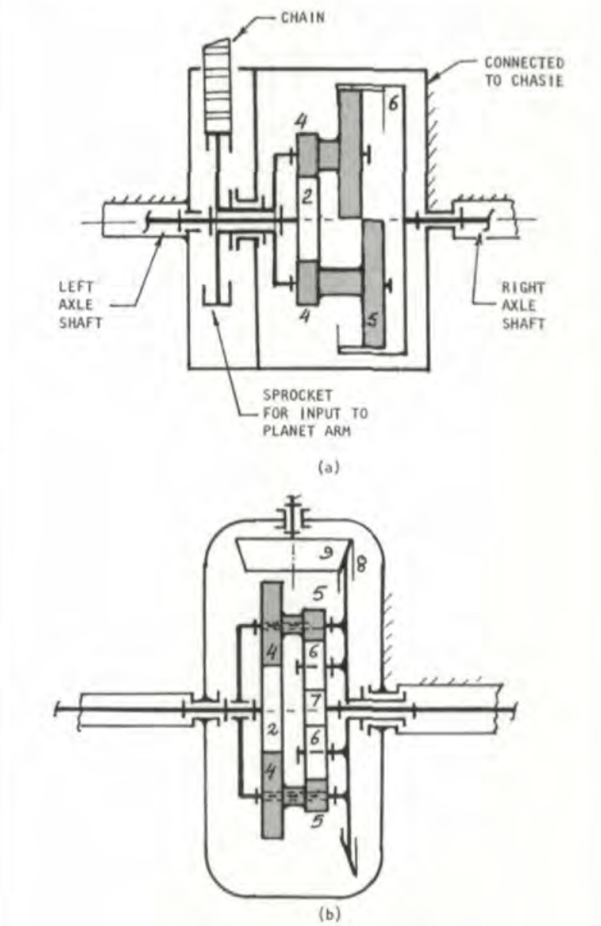

Fig. 15 – Rear-end differentials formed of spur gears, $e = -1$.

are symmetrically mounted for balancing. In (a) chain drives the planet arm; in (b) bevel gears do the same. In (a)

$$-1 = \left(\frac{-N_2}{N_4}\right)\left(\frac{N_5}{N_6}\right) = \left(\frac{-40}{20}\right)\left(\frac{60}{120}\right)$$

In (b)

$$-1 = \left(\frac{-N_2}{N_4}\right)\left(\frac{-N_5}{N_6}\right)\left(\frac{-N_7}{N_8}\right) = \left(\frac{-60}{60}\right)\left(\frac{-20}{40}\right)\left(\frac{-40}{20}\right)$$

Differentials for fractional values of e, such as e = −0.8 for a race car to rotate the outer wheel less than the inner wheel, are used in order to generate stabilizing traction torque that tends to retain the vehicle on the straight line. In that case, e = −0.8 should be satisfied instead of −1 as done above.

Automatic Transmissions

An automatic vehicle transmission in general has two or more simple planetary gear trains whose shafts can be coupled or held stationary to form different arrangements. Input is supplied through one or two fluid couplings (torque converters). Fig. 16 shows a Hydramatic 4L60 (THM 700-R4) automatic transmission by General Motors that generates two forward and one reverse speed reductions. The skeleton of a five-speed automatic transmission with three simple planetary gear trains driven with two torque converters is shown in Fig. 17. It has three clutches at G, H, and P; four sprag overrunning clutches at J, I, N and O.

Observe its operation:

Neutral position. Small converter B-C is empty, gear 3 is held stationary and clutches G and H are open. The planet

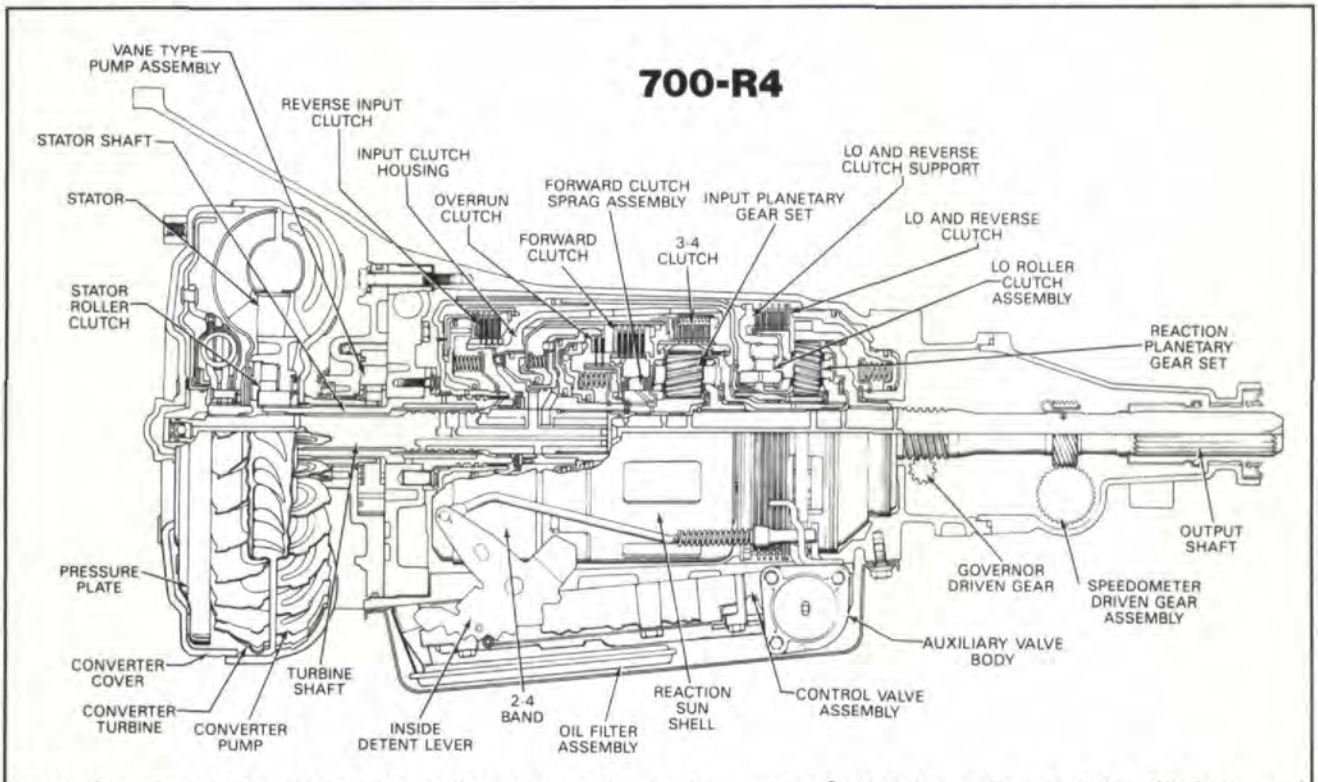

Fig. 16 – Cross section of General Motors 4L60 (THM 700-R4) model automatic transmission.

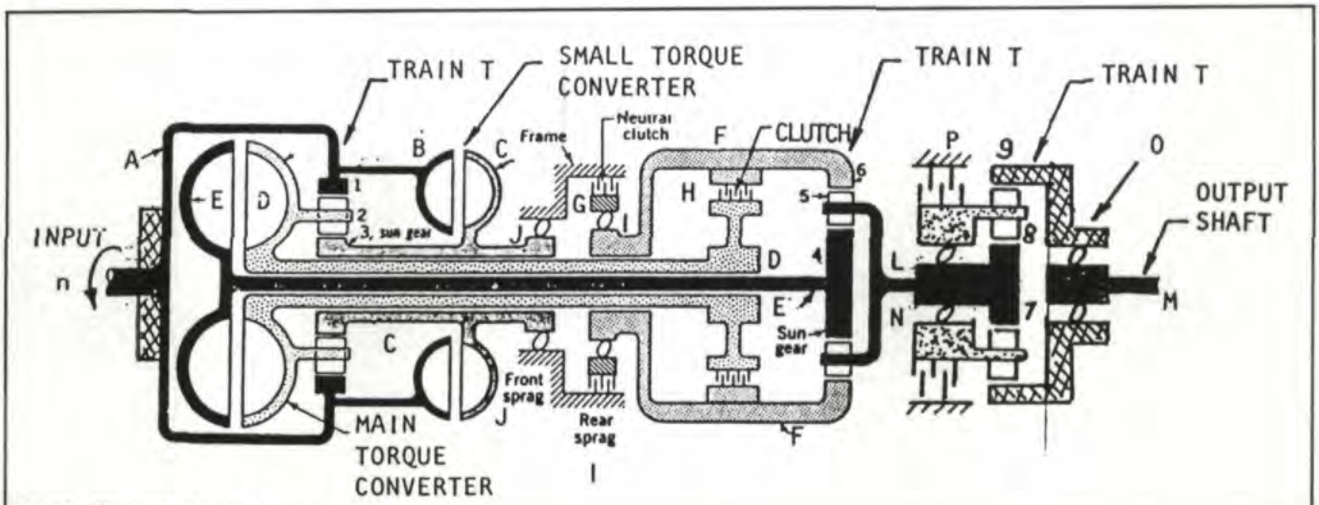

Fig. 17 – Diagram of 5-speed automatic transmission.

arm D rotates freely, $n_4=n_D=n_6$, and shaft L idles, and $n_L=n_6$. P and O are open, N is engaged, $n_7=n_9$ and $n_M=0$.

Low-Low Forward Reduction. B-C is empty, D rotates E and gear 4. Clutches G, J, I retain $n_3=n_6=0$. P is open; N, O are engaged. From train 1

$$e_1 = -\frac{N_1}{N_3} = \frac{0-n_D}{n_1-n_D}, \quad n_D = \frac{e_1}{e_1-1} n_1 \qquad (a)$$

for $e_1=-1.3$ and $n_D=0.5833n_1$.
In train 2, $n_4 = n_D$, and

$$e_2 = -\frac{N_4}{N_6} = \frac{0-n_L}{n_4-n_L} \qquad (b)$$

with $e_2 = -0.7143$ and $n_M=n_L=0.243n_1$.

Second Forward Reduction. H, E are open, G, I are engaged and $n_6=0$. Small converter B-C is full driving gear 3, $n_1=n_3=n_4=n_D$; P is open, N, O are engaged, and from (b) above

$$n_M = n_L = \frac{e_2}{e_2-1} n_1 = 0.4167n_1$$

Third Forward Reduction. J is engaged and $n_3=0$. G and H are engaged, I is open, B-C is empty, P is open, N, O are engaged, and $n_6=n_D=n_E$. Then from (a) above

$$n_M = n_L = n_D = 0.5833n_1$$

Fourth Forward Speed. In this case the entire gear train rotates as one body at the engine speed n_1. I, J, P are open; G, H, N, O are engaged, B-C is full and $n_M = n_L = n_1$.

Reverse Reduction. B-C is full and the system is engaged as for the fourth forward speed, except that P and O are engaged and N is open. From the train value of the third gear train with $n_a=0$,

$$e_3 = -\frac{N_7}{N_9} = \frac{n_M}{n_1}$$

with

$$e_3 = -1/3, \quad n_M = -n_1/3$$

Conclusions

Offered in the foregoing with illustrative examples are efficient methods of analysis and synthesis of compound planetary gear trains and planetary differentials. The methods use the train values of component planetary gear trains, and lead to very simple design processes. As expected, by introducing worm-gear sets in a compound planetary gear train, very large speed reductions can be generated with fewer number of gears in the train.

References

1. PLANTARY GEAR REDUCERS, Crichton Manufacturing Co., Johnstown, PA.
2. MACHINE DESIGN, Vol. 54, No. 18, August 12, 1982, p.55.