Determination of Gear Ratios

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Introduction

Selection of the number of teeth for each gear in a gear train such that the output to input angular velocity ratio is a specified value is a problem considered by relatively few published works on gear design. McComb and Matson [1] have listed five methods, all of which involve cut-and-try procedures, Spotts [2] described a sixth cut-and-try technique, and Page [3] listed 14,000 gear ratios and the number of teeth on the gears involved.

The following sections show that several theorems from continued, or chain, fractions are applicable to the gear problem at hand, and suggest a direct means for determining the number of teeth on each gear. A numerical method is outlined and three examples of its use terminate the discussion.

Continued Fractions

Following Hardy and Wright [4], the continued fraction

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_N}}}} \quad (1)$$

will be denoted by $[a_0, a_1, \ldots, a_N]$ if the continued fraction has $N + 1$ variables $a_0, a_1, \ldots, a_N$ and by $[a_0, a_1, \ldots]$ if it continues indefinitely. If $0 \leqq n \leqq N$ the partial fraction represented by $[a_0, a_1, \ldots, a_n]$ is termed the nth *convergent* to $[a_0, a_1, \ldots a_N]$. A continued fraction is said to be *simple* if a_1 through a_N inclusive are all positive and integral.

The fundemental theorem for gear applications is that *Theorem 1*:

If p_n and q_n are defined by

$$p_0 = a_0 q_0 = 1$$

$$p_1 = a_0 a_1 + 1 \quad q_1 = a_1 \quad (2)$$

$$p_n = a_n p_{n-1} + p_{n-2} \quad (3)$$

$$q_n' = a_n q_{n-1} + q_{n-2} \quad (4)$$

then

$$[a_0, a_1, \ldots, a_n] = \frac{p_n}{q_n}. \quad (5)$$

Substitution into (1) verifies that

$$[a_0] = a_0 \quad (6)$$

$$[a_0, a_1] = \frac{a_0 a_1 + 1}{a_1} \quad (7)$$

$$[a_0, a_1, a_2] = \frac{a_2 a_1 a_0 + a_2 + a_0}{a_2 a_1 + 1} \quad (8)$$

from which it follows, after algebraic manipulation, that

$$[a_0, a_1] = a_0 + \frac{1}{a_1} \quad (9)$$

and

$$[a_0, a_1, \ldots, a_{n-1}, a_n] = [a_0, a_1, \ldots, a_{n-2}, a_{n-1} + \frac{1}{a_n}] \quad (10)$$

Equations (6) and (7) verify the theorem for $n = 0$ and $n = 1$. Assuming it to be true for $n \leqq m$, for $m < N$, then

$$[a_0, a_1, \ldots, a_{m-1}, a_m] = \frac{p_m}{q_m} = \frac{a_m p_{m-1} + p_{m-2}}{a_m q_{m-1} + q_{m-2}} \quad (11)$$

where p_{m-1}, p_{m-2}, q_{m-1}, q_{m-2} all depend only upon a_0, $a_1, \ldots, a_{m-1}$. From (10) it follows that

$$[a_0, a_1, \ldots, a_{m-1}, a_m, a_{m+1}]$$

$$= \left[a_0, a_1, \ldots, a_{m-1}, a_m + \frac{1}{a_{m+1}}\right]$$

$$= \frac{\left(a_m + \frac{1}{a_{m+1}}\right) p_{m-1} + p_{m-2}}{\left(a_m + \frac{1}{a_{m+1}}\right) q_{m-1} + q_{m-2}}$$

$$= \frac{a_{m+1}(a_m p_{m-1} + p_{m-2}) + p_{m-1}}{a_{m+1}(a_m q_{m-1} + q_{m-2}) + q_{m-1}}$$

$$= \frac{a_{m+1} p_m + p_{m-1}}{a_{m+1} q_m + q_{m-1}} = \frac{p_{m+1}}{q_{m+1}}. \quad (12)$$

and the theorem is proved by induction. Replace m by $n - 1$ in (12) to obtain relations (3) and (4).

Hardy and Wright [4] also proved the following theorems which are of importance to gear design.

Theorem 2:
The convergents to a simple continued fraction are in their lowest terms.

Theorem 3:
Any rational number can be represented by a finite simple continued fraction.

Theorem 4:
If $N > 1$, $n > 0$, then the difference

$$\frac{p_N}{q_N} - \frac{p_n}{q_n}$$

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decreases steadily in absolute value as n increases. Also

$$q_n \frac{p_N}{q_N} - p_n = \frac{(-1)^n \delta_n}{q_{n+1}}$$

where $0 < \delta_n < 1, \delta_{N-1} = 1$ for $1 \leqq n \leqq N-2$

and

$$\left| \frac{p_N}{q_N} - \frac{p_n}{q_n} \right| \leqq \frac{1}{q_n q_{n-1}} \leqq \frac{1}{q_n^2}$$

for $n \leqq N-1$, with inequality in both places except when $n = N-1$.

Theorem 5:

If $a_0, a_1, a_2, \ldots$ is a sequence of integers for which $a_1, a_2, \ldots$ are all positive, then $[a_0, a_1, \ldots, a_n]$ tends to a limit as n goes to infinity.

Theorem 6:

Every irrational number can be expressed in just one way as an infinite simple continued fraction.

Theorem 7:

Theorem 4 holds (except for references to N) for infinite continued fractions, and

$$\left| \frac{p_N}{q_N} - \frac{p_n}{q_n} \right| < \frac{1}{q_n q_{n+1}} < \frac{1}{q_n^2}.$$

Together these mean that the numerator and denominator of the rational gear ratio have no factors in common and that if the gear ratio is irrational it is possible to approximate it to within any degree of accuracy by increasing the number of terms in the continued fraction. In particular, this enables a computer program to be written in which the user may specify, and realize, the accuracy of a gear train to approximate any desired speed ratio.

Numerical Method

Evaluation of the $a_0, a_1, \ldots$ terms is according to Euclid's algorithm, which yields

$$a_1 = \mathrm{INT}(R) \qquad y_1 = \mathrm{FRC}(R)$$
$$p_1 = a_1 \qquad q_1 = 1 \tag{13}$$

$$a_2 = \mathrm{INT}\left(\frac{1}{y_1}\right) \qquad y_2 = 1 - a_2 y_1$$
$$p_2 = a_1 a_2 + 1 \qquad q_2 = a_2 \tag{14}$$

and for $n \geqq 3$,

$$a_n = \mathrm{INT}\left(\frac{y_{n-2}}{y_{n-1}}\right) \qquad y_n = -a_n y_{n-1} + y_{n-2}$$
$$p_n = a_n p_{n-1} + p_{n-2} \qquad q_n = a_n q_{n-1} + q_{n-2} \tag{15}$$

where R is either the desired ratio of the angular velocity of the output shaft to that of the input shaft, or the inverse, INT (R) is the largest integer equal to or less than R, and FRC (R) = R − INT (R).

Because of the variety of programming codes, the program will be presented as a flow chart. In it E is the maximum acceptable error, defined as

$$E = \left| R - \frac{p_N}{q_N} \right|$$

Input data to program GEAR RATIO is the desired speed ratio, R, and the magnitude of the acceptable error, E. Error E may be decreased by REVISE if the user wished to examine the effect of considering rational approximations for a sequence of decreasing E values.

Examples

Three examples will be considered: one where R is rational and p_N and q_N are factorable, one where both p_N and q_N are not factorable, and one where R is irrational.

Example 1. Find a gear train providing an angular velocity ratio of 2.871 with an error not greater than 0.000001.

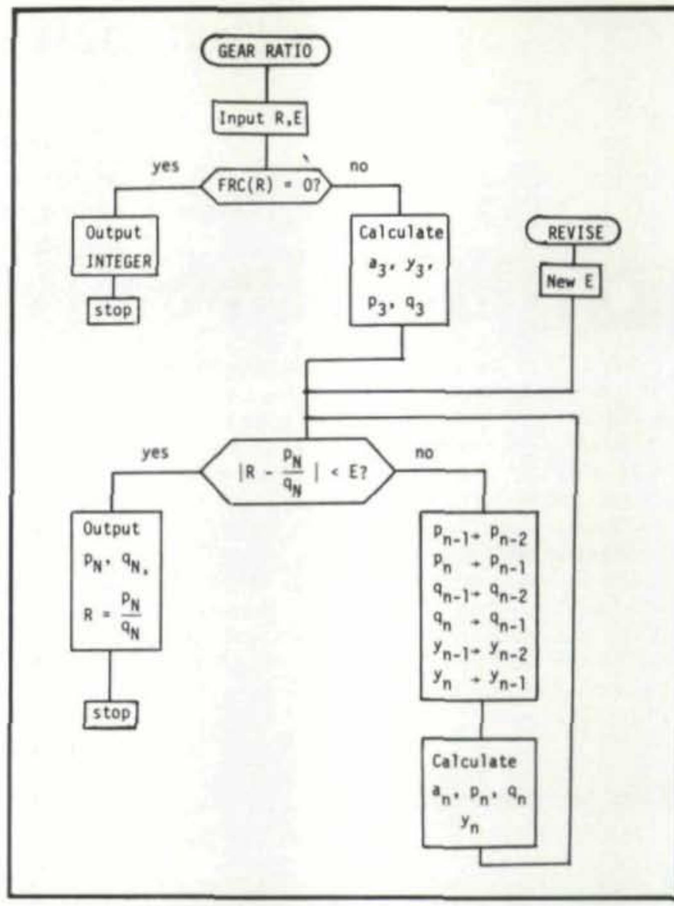

FIG. 1—Flowchart for GEAR RATIO program

Computation time for a program implemented on a Hewlett-Packard HP-41CV is approximately 9.9 s to find that

$$\frac{p_N}{q_N} = \frac{2871}{1000}$$

is the only rational that will satisfy the accuracy requirement. Use of a factoring routine leads to the possible gear train

$$R = \frac{\omega_1}{\omega_2} = \frac{33}{25} \times \frac{87}{40} = 2.871000.$$

Thus, $N_1 = 25$, $N_2 = 33$, $N_3 = 40$, and $N_4 = 87$, where gears 2 and 3 are on the same shaft, gear 1 on the input shaft meshes with gear 2, and gear 4 on the output shaft meshes with gear 3.

Example 2. Find a gear train providing an angular velocity ratio of 2.68 with an error no more than 0.0001.

According to formulas (13), (14), and (15)

$$a_1 = \mathrm{INT}(2.68) = 2 \qquad a_2 = \mathrm{INT}\left(\frac{1}{y_1}\right) = 1$$
$$p_1 = 2 \qquad p_2 = 3$$
$$q_1 = 1 \qquad q_2 = 1$$
$$y_1 = 0.68 \qquad y_2 = 1 - 0.68 = 0.32$$

$$a_3 = \mathrm{INT}\left(\frac{0.68}{0.32}\right) = 2 \qquad a_4 = 8$$
$$p_3 = 8 \qquad p_4 = 67$$
$$q_3 = 3 \qquad q_4 = 25$$
$$y_3 = 0.04 \qquad y_4 = 0$$

so that $N_2 = p_4 = 67$, $N_1 = q_4 = 25$, and $R = 67/25 = 2.680000$.

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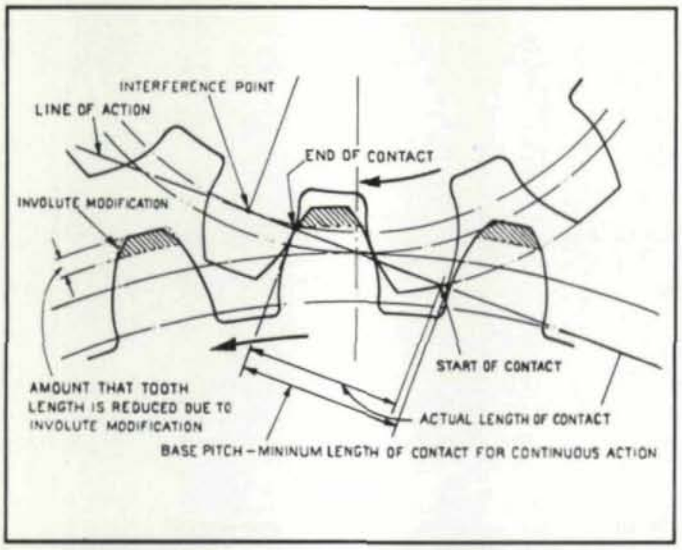

Fig. 19 — Diagram Illustrating Lack of Continuous Action Resulting from Involute Modification.

interrupted action. Fig. 17 shows diagrammatically a pair of gears in which the teeth are shortened to such an extent that they lack continuous action, and hence fail to transmit uniform angular motion. It will be noticed that the base pitch, which is the circular pitch transferred to the base circle is greater in length than the actual length of contact.

Lack of Continuous Action Due to Undercut

Fig. 18 shows another condition that can result in lack of continuous action. Here two 12-tooth pinions are shown in mesh, and it will be seen that the actual length of involute contact falls far short of meeting the requirements for continuous action. There are two reasons for this. One is the small number of teeth in the mating gears, and the other is the small pressure angle. The small pressure angle results in a severe undercutting of the flanks of the teeth, and the small number of teeth, of course, reduces the number of teeth in contact.

Lack of Continuous Action Due to Tooth Modification

Another condition that can cause lack of continuous action is involute modification. Fig. 19 illustrates a pair of gears, the tooth shape on one of these gears being modified near the tip. In this case, true involute contact does not start at the tip of the tooth on one gear, and hence, falls short of complete profile contact. Under light loading conditions, these gears might fail for continuous action, because of the shortened tooth profile above the pitch circle; but under heavy loading, the contact could extend over a greater length of the tooth profile. Gears are sometimes cut "off" pressure angle to compensate for tooth deflection under heavily loaded conditions; also to take care of tooth distortion due to heat treatment.

DETERMINATION OF GEAR RATIOS . . .

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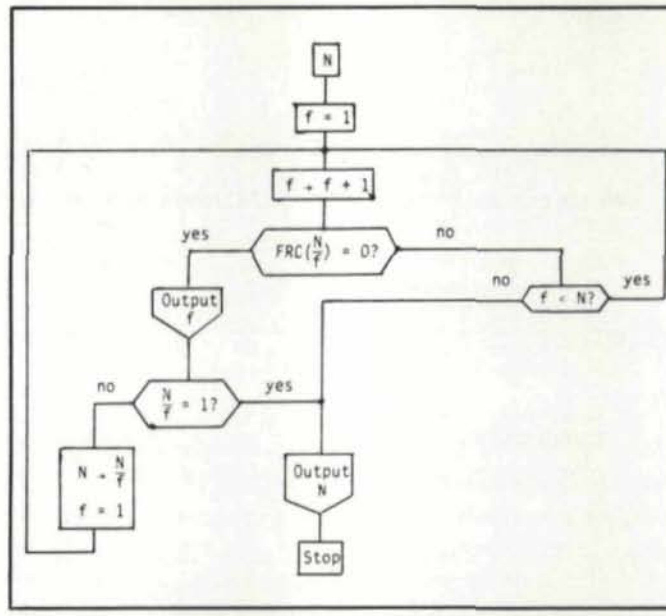

FIG. 2—Flowchart for FACTOR program

Example 3. Design a gear train to provide an angular velocity ratio of $\sqrt{5}$. Computer output is displayed in Table 1 along with the computation times using program GEAR RATIO followed by successive uses of REVISE.

Use of a factoring routine once again for $N_1 = 2889$ and $N_2 = 1292$ yields a four gear set with

$$R = \frac{27}{19} \times \frac{107}{68} = 2.2360681.$$

Thus, $N_1 = 19$, $N_2 = 27$, $N_3 = 68$, and $N_4 = 107$ teeth. Obtaining still greater accuracy is more difficult from a practical viewpoint because 13 and 421 are the only factors of 5473 while 47 and 1103 are the only factors of 51,841.

The flowchart for a typical factoring routine is shown in Fig. 2.

Table 1 Rational numbers approximating $\sqrt{5}$

E	p_N	q_N	Approx. computation* time (s)
1×10^{-2}	38	17	4
1×10^{-3}	38	17	2
1×10^{-4}	161	72	3
1×10^{-5}	682	305	4
1×10^{-6}	2889	1292	3
1×10^{-7}	12,238	5473	4
1×10^{-8}	12,238	5473	5
1×10^{-9}	115,920	51,841	5

* The first calculation used the GEAR RATIO program and the remaining calculations were performed by entering the program at REVISE. Direct calculation of the last ratio (115 920/51841) using GEAR RATIO required approximately 18 s, rather than the 30 required for the step-by-step calculations.

References

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