Determining the Shaper Cut Helical Gear Fillet Profile

George Lian

Management Summary

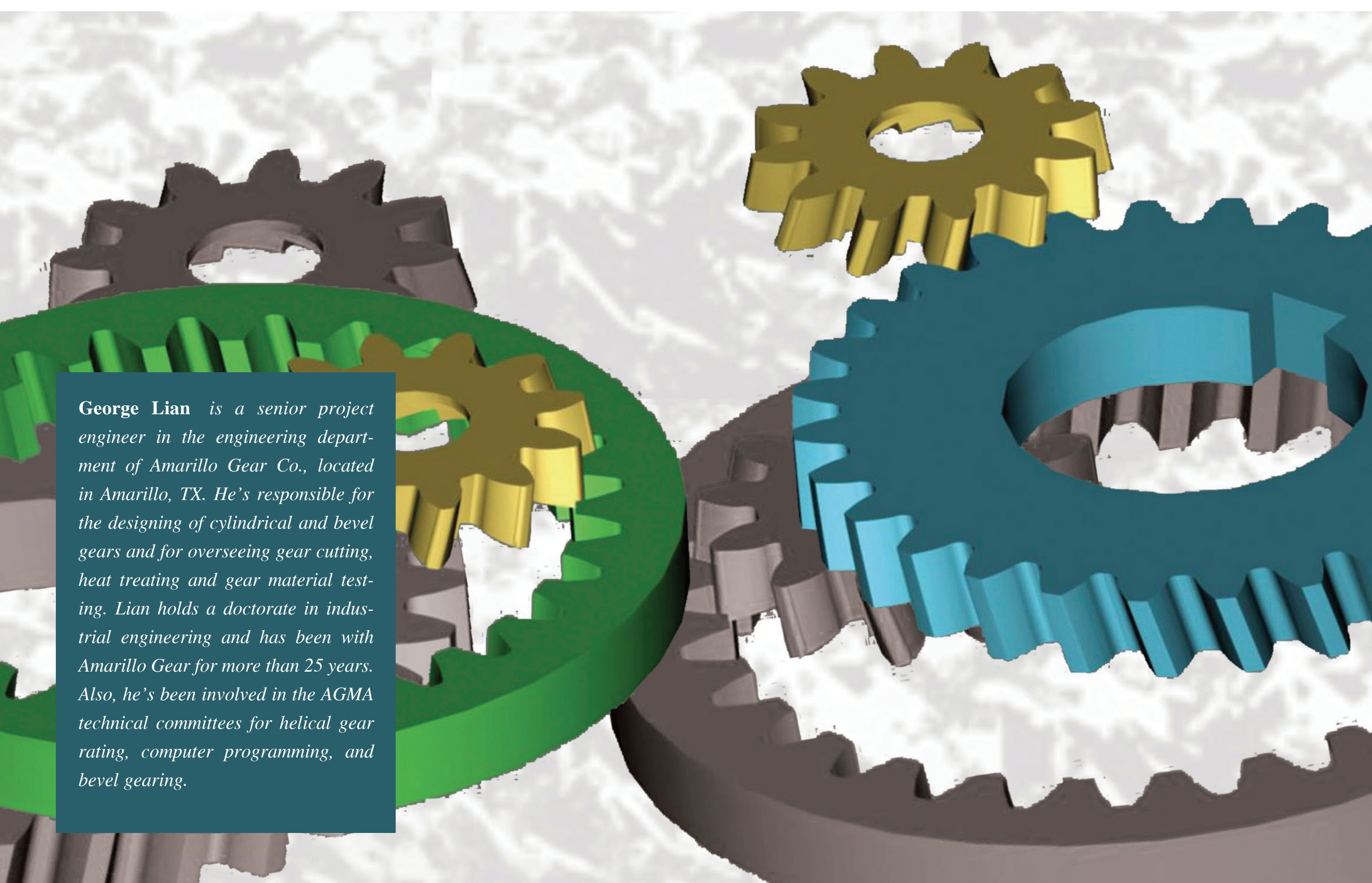
This article describes a root fillet form calculating method for a helical gear generated with a shaper cutter. The shaper cutter considered has an involute main profile and an elliptical cutter edge in the transverse plane. Since the fillet profile cannot be determined with closed-form equations, a Newton's approximation method was used in the calculation procedure. The article also explores the feasibility of using a shaper tool algorithm for approximating a hobbed fillet form. Finally, the article discusses some of the applications of fillet-form calculation procedures, such as form diameter (start of involute) calculation and finishing stock analysis.

Introduction

Analytical methods for determining the gear fillet profile (trochoid) have been well documented. Khiralla (Ref. 1) described methods for calculating the fillet profile of hobbed and shaped spur gears. Colbourne (Ref. 2) provided equations for calculating the trochoid of both involute and non-involute gears generated by rack or shaper tools. The MAAG Gear Handbook (Ref. 3) also provided equations for calculating trochoids generated with rack-type tools that have circular tool tips. Vijayakar, et al. (Ref. 4) presented a method of determining spur gear tooth profiles using an arbitrary rack. The above mentioned are only samples of many published works. However, the method for determining the trochoid of a helical gear generated with a shaper tool is not widely published. This article presents an intuitive algorithm where the fillet profile of a shaper-tool-generated external or internal helical gear can be calculated.

A shaper tool generating a gear can be visualized as a gear set meshing with zero backlash. The algorithm in this article is based on a shaper tool in tight mesh with a semi-finished helical gear. The semi-finished gear geometry was used for calculation because the shaper tool used as the semi-finishing tool is usually the one that generates the trochoid. However, if the shaper cutter *is* the finishing tool, the algorithm presented will also work by letting the finishing stock equal zero. The trochoid of a spur gear can also be calculated by letting the helix angle equal zero.

The shaper tool used in this algorithm may have a different reference normal pressure angle than that of the gear. A necessary condition for a shaper tool to generate the correct involute profile on a gear is that both the tool and the gear must have



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equal normal base pitches. This article stipulates that the axis of the shaper tool and the gear are parallel, which is often true for gear shaping. Consequently, the shaper tool and the gear must also have equal base helix angles.

Although the algorithm is based on the shaper cutter as a generating tool, the presented method can also be used to calculate a trochoid generated with a hob or a rack-type tool if the number of the shaper teeth is large (e.g. 10,000).

Symbols and Conventions

The symbols are defined where first used. This article tries to adhere to the following rules in subscript usage:

- Symbols related to tool geometry have subscript "0".
- No subscript is used for symbols related to the gear.
- Subscript "n" is used for measurements in the normal plane.
- Subscript "r" is used for symbols related to the semi-finished gear.
- Subscript "g" is used for symbols related to the generating pitch circle.

When dual signs are used in an equation (e.g. ±), the upper sign is for external gears and the lower one for internal gears.

Non-italicized uppercase symbols are used to designate points on the shaper tool, the gear, or other points of interest. Points are also represented as the coordinates (x,y). The length of a vector (e.g. R) is represented as $\|R\|$.

Coordinate System

The reference position of a shaper tool generating a gear is depicted in Figure 1 for external gear shaping and Figure 2 for internal.

The following coordinate system and sign conventions are followed:

- A standard cartesian coordinate system is used. The center of the shaper tool O_0 is (0,0).
- The reference position of the shaper tool is with one of its teeth aligned with the y-axis. The end of the shaper tooth points in the –y direction.
- The center of the gear, O_G, is also on the y-axis with one of the tooth spaces aligned with the y-axis. The opening of the tooth space is in the +y direction.
- Angular measures, related to tool or gear rotation or location of a point, have signs. Counterclockwise rotation from the reference line is positive, and clockwise is negative.

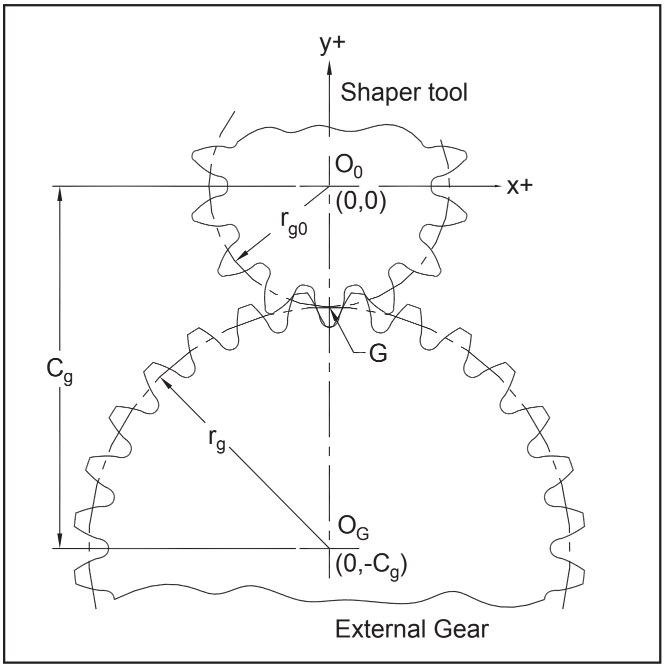

Figure 1—Shaping an external gear.

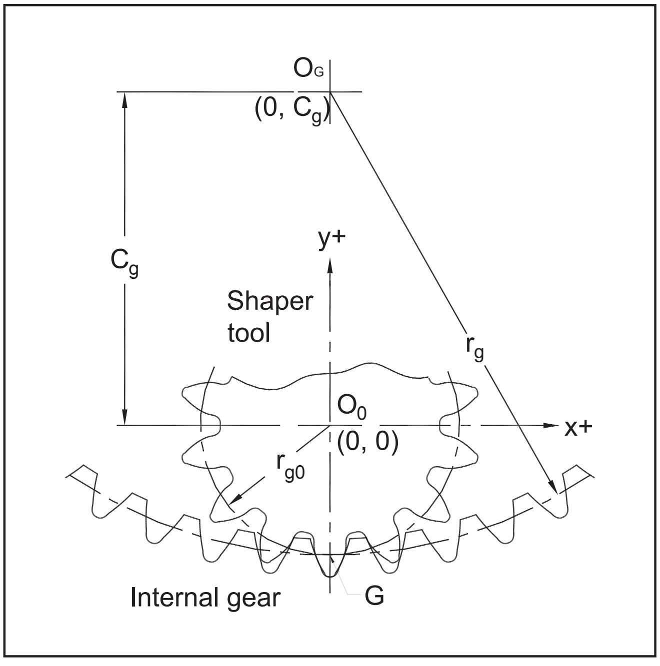

Figure 2—Shaping an internal gear.

Shaper Tool and Gear Geometry

The following are required tool and gear data for calculating the trochoid:

Shaper tool data:

P_{nd0} is the reference normal diametral pitch, tool (in.$^{-1}$)
n_0 is the number of teeth, tool
ϕ_{n0} is the reference normal pressure angle, tool
ψ_0 is the reference helix angle, tool
s_{n0} is the reference normal circular thickness, tool (in.)
d_{a0} is the outside diameter, tool (in.)
ρ_0 is the tool tip radius (in.)
δ_0 is the protuberance (in.)

Gear data:

P_{nd} is the reference normal diametral pitch, gear (in.$^{-1}$)
n is the number of teeth, gear
ϕ_n is the reference normal pressure angle, gear
ψ is the reference helix angle, gear
s_n is the reference normal circular thickness, gear (in.)
μ_s is the stock allowance per flank, gear (in.), defined on the reference pitch circle (not along the base tangent).

Basic Shaper Tool and Gear Geometry

The following equations calculate the basic tool and gear

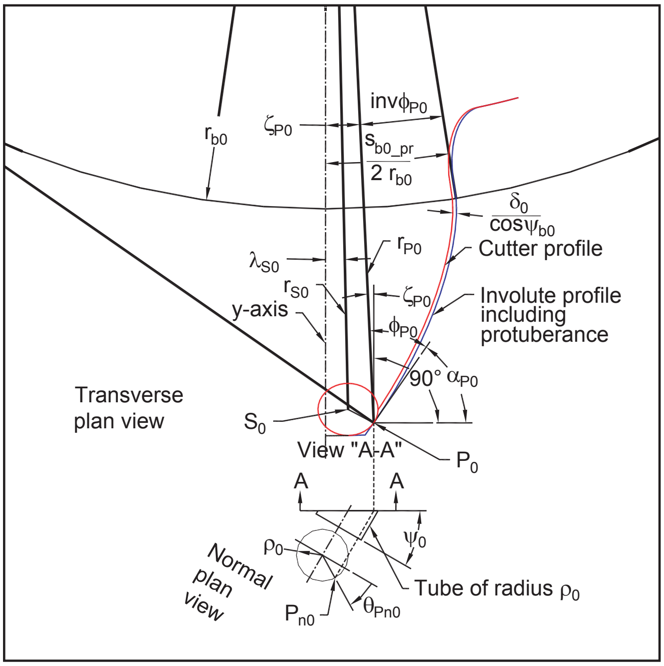

Figure 3—Tool tip of a shaper tool.

geometry:

Standard transverse pressure angle of tool, ϕ_0

$$\phi_0 = \arctan\left(\frac{\tan\phi_{n0}}{\cos\psi_0}\right) \tag{1}$$

Standard reference pitch radius of tool, r_0 (in.)

$$r_0 = \frac{n_0}{2P_{nd0}\cos\psi_0} \tag{2}$$

Base radius of tool, r_{b0} (in.)

$$r_{b0} = r_0\cos\phi_0 \tag{3}$$

Reference transverse circular thickness of tool, S_0 (in.)

$$s_0 = \frac{s_{n0}}{\cos\psi_0} \tag{4}$$

Transverse base pitch of tool, p_{b0} (in.)

$$p_{b0} = \frac{2\pi r_{b0}}{n_0} \tag{5}$$

Normal base pitch of tool, p_{nb0} (in.)

$$p_{nb0} = \frac{\pi\cos\phi_{n0}}{P_{nd0}} \tag{6}$$

Base helix angle of tool, ψ_{b0}

$$\psi_{b0} = \arccos\left(\frac{p_{nb0}}{p_{b0}}\right) \tag{7}$$

Base circular thickness of tool, s_{b0} (in.)

$$s_{b0} = 2r_{b0}\left(\frac{s_0}{2r_0} + \mathrm{inv}\phi_0\right) \tag{8}$$

where

inv is the involute function of an angle
inv $\alpha = \tan\alpha - \alpha$

Standard reference pitch radius of gear, r (in.)

$$r = \frac{n}{2P_{nd}\cos\psi} \tag{9}$$

Base radius of semi-finished gear, r_{br} (in.)

$$r_{br} = r_{b0}\frac{n}{n_0} \tag{10}$$

The helix angle at standard pitch radius of semi-finished gear, ψ_r

$$\psi_r = \arctan\left(\frac{r\tan\psi_{b0}}{r_{br}}\right) \tag{11}$$

Transverse pressure angle at reference pitch radius of semi-finished gear, ϕ_r

$$\phi_r = \arccos\left(\frac{r_{br}}{r}\right) \tag{12}$$

Transverse circular thickness of semi-finished gear, s_r (in.)

$$s_r = \frac{s_n + 2\mu_s}{\cos\psi_r} \tag{13}$$

Base circular thickness of semi-finished gear, s_{br} (in.)

$$s_{br} = 2r_{br}\left(\frac{s_r}{2r} \pm \mathrm{inv}\,\phi_r\right) \tag{14}$$

Center of Tool Tip on a Shaper Tool

A shaper tool for gear semi-finishing usually has protuberance. It generates undercut on a gear, so that the finishing tool only needs to machine the involute profile of the gear. To obtain the designed amount of protuberance on a shaper tool, the tool tip is made tangent to the involute profile that is temporarily formed by increasing the shaper tooth thickness to include the protuberance (see Fig. 3). The tangent point, common to the tool tip and the involute profile, will be referred to as the profile tangent point, P_0 . When the temporarily formed involute profile is removed, the shaper tool will have the designed amount of protuberance.

The shaper tool tip is also made tangent to the outside diameter of the tool (see Fig. 4) so that the transition from the outside diameter to the tool tip will be smooth. The common tangent point on the shaper tool tip and the outside diameter of the tool will be referred to as the end tangent point, E_0.

The following are the required data for calculating the center of the shaper tool tip:

d_{a0} is the outside diameter, tool (in.)
s_{b0} is the base circular thickness, tool (in.)
ρ_0 is the tool tip radius (in.)
δ_0 is the protuberance (in.)
ψ_{b0} is the base helix angle, tool
ψ_0 is the reference helix angle, tool

The base circular thickness of the involute profile, formed by increasing the shaper tool tooth thickness to include the protuberance, s_{b0_pr}

$$s_{b0\_pr} = s_{b0} + \frac{2\delta_0}{\cos\psi_{b0}} \tag{15}$$

Coordinates of the center of the tool tip, S_0

$$S_0 = (r_{S0}\sin\lambda_{S0} , -r_{S0}\cos\lambda_{S0}) \tag{16}$$

where

r_{S0} is the tool radius to the center of the tool tip (in.)

λ_{S0} is the offset angle of the tool tip. For a shaper tool with full tip radius, λ_{S0} will equal zero.

Coordinates of the profile tangent point, P_0, are

$$P_0 = S_0 + (\rho_0 \frac{\cos\theta_{Pn0}}{\cos\psi_0}, \rho_0 \sin\theta_{Pn0}) \tag{17}$$

where

θ_{Pn0} is the auxiliary angle that locates P_0. The angle is measured in the normal plane, clockwise from the horizontal axis of the tool tip. θ_{Pn0} will usually have a negative value.

The tool radius to profile tangent point, r_{P0} (in.), is

$$r_{P0} = \| P_0 \| \tag{18}$$

The transverse pressure angle, ϕ_{P0}, at P_0 is

$$\phi_{P0} = \arccos(\frac{r_{b0}}{r_{P0}}) \tag{19}$$

The tangent angle, α_{P0}, at P_0 (the derivation of Equation 20 is given in Appendix A) is

$$\alpha_{P0} = \arctan (\frac{-\cos\psi_0}{\tan\theta_{Pn0}}) \tag{20}$$

The angle between the y-axis and the radius to the profile tangent point, ζ_{P0}, is

$$\zeta_{P0} = \frac{s_{b0\_pr}}{2r_{b0}} - \text{inv}\phi_{P0} \tag{21}$$

The coordinates of the end tangent point, E_0, are

$$E_0 = S_0 + (\rho_0 \frac{\cos\theta_{En0}}{\cos\psi_0} , \rho_0\sin\theta_{En0}) \tag{22}$$

where

θ_{En0} is the auxiliary angle that locates E_0. The angle is measured in the normal plane, clockwise from the horizontal axis of the tool tip. θ_{En0} will usually have a negative value.

The angle of tangent, α_{E0}, at the end tangent point, E_0, is

$$\alpha_{E0} = \arctan (\frac{-\cos\psi_0}{\tan\theta_{En0}}) \tag{23}$$

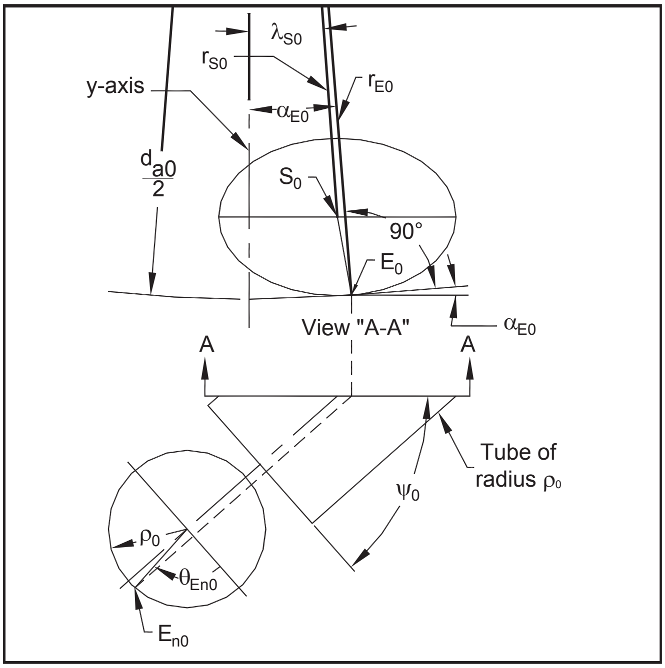

Figure 4—End of tool tip (with helix angle exaggerated).

The tool radius to end tangent point, r_{E0} (in.), is

$$r_{E0} = \| E_0 \| \tag{24}$$

The following are conditions for the tool tip to position properly on a shaper tool tooth:

1) The profile tangent point, P_0, on the tool tip must also be a point on the involute profile that includes the protuberance, thus

$$\alpha_{P0} + \phi_{P0} - \zeta_{P0} - \frac{\pi}{2} = 0 \tag{25}$$

2) The angle, ζ_{P0}, subtended by one half of the transverse circular thickness of the involute curve (include the tool protuberance) at P_0, must equal the angle formed by the y-axis and the line connecting the center of the tool to P_0.

$$\zeta_{P0} - \arcsin (\frac{x_{P0}}{r_{P0}}) = 0 \tag{26}$$

where

x_{P0} is the x-coordinate of profile tangent point, P_0 (in.)

3) The end tangent point must also be a point on the outside diameter of the shaper tool, thus

$$r_{E0} - \frac{d_{a0}}{2} = 0 \tag{27}$$

4) The tangent angle, α_{E0}, at the end tangent point, E_0, must equal the angle formed by the y-axis and the line connecting the center of the tool to E_0

$$\alpha_{E0} - \arcsin (\frac{x_{E0}}{r_{E0}}) = 0 \tag{28}$$

Equations 25–28 must all be satisfied for the tool tip to be correctly positioned on a shaper tool tooth. The variables to be determined are r_{S0}, λ_{S0}, θ_{Pn0} and θ_{En0}. Since the systems of the equations are transcendental and cannot be solved directly, the Newton's method is used to calculate the roots for Equations 25–28.

Solving the System of Non-linear Equations for Center of Tool Tip

For simplicity, rewrite Equations 25–28 as generic vector equations in the form

$$F(X) = 0 \qquad (29)$$

where

$$F(X) = (f_1(X), f_2(X), f_3(X), f_4(X))^T = (\text{Eq. 25, Eq. 26, Eq. 27, Eq. 28})^T \qquad (30)$$

$$0 = (0,0,0,0)^T \qquad (31)$$

$$X = (x_1, x_2, x_3, x_4)^T = (r_{S0}, \lambda_{S0}, \theta_{Pn0}, \theta_{En0})^T \qquad (32)$$

The Newton's iteration equation (Ref. 6) is written as

$$X1 = X + \delta X \qquad (33)$$

where δX satisfies the following system of linear equations

$$J \bullet \delta X = -F(X) \qquad (34)$$

where

X1 is the vector of the new roots for the next iteration
X is the vector of current roots
δX is the vector of Newton's steps for the next iteration
J is the Jacobian matrix

where

$$J = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_4} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_4}{\partial x_1} & \cdots & \cdots & \frac{\partial f_4}{\partial x_4} \end{pmatrix} \qquad (35)$$

$\frac{\partial f_i}{\partial x_j}$ is the partial derivative of the i^{th} equation with respect to the j^{th} variable

The partial derivatives in the Jacobian matrix can be approximated using the finite differences

$$\frac{\partial f_i}{\partial x_j} \approx \frac{f_i(X + \Delta X_j) - f_i(X)}{\delta x_j} \qquad (36)$$

where

i is the i^{th} row of the Jacobian matrix
j is the j^{th} column of the Jacobian matrix
ΔX_j is a vector with its j^{th} element equal to the j^{th} element of the current Newton's step, δX, and all remaining elements equal 0

For each iteration, the sum of the absolute values of the functions (errors) is calculated.

$$ERR(X1) = \sum_{i=1}^{4} |f_i(X1)| \qquad (37)$$

The Newton's iteration procedure is terminated when the error (see Eq. 37) becomes smaller than a predetermined tolerance, or when a predetermined number of iterations has been reached.

The Newton's iteration procedure is described below:

1) Select a set of initial guess values for the new root, X1. The following are the suggested values:

$$x_1(r_{S0}) = \frac{d_{a0}}{2} - \rho_0$$
$$x_2(\lambda_{S0}) = 0.0175$$
$$x_3(\theta_{Pn0}) = -\phi_{n0}$$
$$x_4(\theta_{En0}) = -1.4835$$

2) Select the initial Newton's steps, δX. The following values work satisfactorily:

$$\delta X = (0.01, 0.01, 0.01, 0.01)^T$$

3) Evaluate the system of non-linear equations (see Eq. 30) at the new root, F(X1).
4) Calculate the error ERR(X1) (see Eq. 37).
5) The iteration is terminated, if ERR(X1) $\leq 10^{-10}$ or if a predetermined number of iterations (30 should be sufficient) have been reached. Otherwise, continue with the next step.
6) Save the new roots as the current roots, so that a new set of roots can be calculated

$$X = X1 \qquad (38)$$

7) Calculate the Jacobian matrix, column by column, starting with column one using Equation 36. Repeat the calculation procedure for the remaining columns until the Jacobian matrix is completed (see Eq. 35).
8) Solve the system of linear equations (see Eq. 34) for the next set of the Newton's steps, δX.
9) Calculate new roots, X1, using Equation 33.
10) Repeat steps 3–9 until step 5 is satisfied.

The system of linear equations in step 8 (see Eq. 34) can be solved by inverting the Jacobian matrix or by using one of many numerical root finding algorithms, such as Gaussian elimination method (Ref. 7).

Generating Pressure Angle and Center Distance

The generating pressure angle and the center distance are based on tight meshing of a shaper tool with a semi-finished gear. The involute function of the generating pressure angle, $\text{inv}\phi_g$, is given by the following equation (the derivation of Equation 39 is given in Appendix A):

$$\text{inv}\phi_g = \frac{s_{b0} + s_{br} - p_{b0}}{2(r_{b0} \pm r_{br})} \tag{39}$$

where

- s_{b0} is the base circular thickness of the tool (in.)
- s_{br} is the base circular thickness of the semi-finished gear (in.)
- p_{b0} is the transverse base pitch of the tool (in.)
- r_{b0} is the base radius of the tool (in.)
- r_{br} is the base radius of the semi-finished gear (in.)

The generating pressure angle, ϕ_g, can be calculated by taking the arc of the involute function (Ref. 5). The generating center distance, c_g (in.), is

$$c_g = \frac{r_{br} \pm r_{b0}}{\cos\phi_g} \tag{40}$$

The generating pitch radius of the shaper tool, r_{g0}, is

$$r_{g0} = \frac{r_{b0}}{\cos\phi_g} \tag{41}$$

The generating pitch radius of the gear, r_g, is

$$r_g = r_{g0}\frac{n}{n_0} \tag{42}$$

Determination of Shaper-Tool-Generated Fillet Profile

Conjugate point of an arbitrary point on a shaper tool tip. The fillet profile (trochoid) of a helical gear is generated by the tool tip of a shaper tool. This section describes the procedure for calculating a point on the trochoid that is conjugate to an arbitrary point on the shaper tool tip, X_0 (see Fig. 5).

The coordinates of an arbitrary point, X_0, on the tool tip are

$$X_0 = S_0 + (\rho_0 \frac{\cos\theta_{Xn0}}{\cos\psi_0}, \rho_0\sin\theta_{Xn0}) \tag{43}$$

where

- S_0 are the coordinates of the center of tool tip (in., in.)
- ρ_0 is the tool edge radius (in.)
- θ_{Xn0} is the auxiliary angle that locates an arbitrary point on the tool tip. This angle is measured in the normal plane, clockwise from the horizontal axis of the tool tip. θ_{Xn0} will usually have a negative value.

The slope, m_{X0}, of the normal passing through X_0 is

$$m_{X0} = \frac{\tan\theta_{Xn0}}{\cos\psi_0} \tag{44}$$

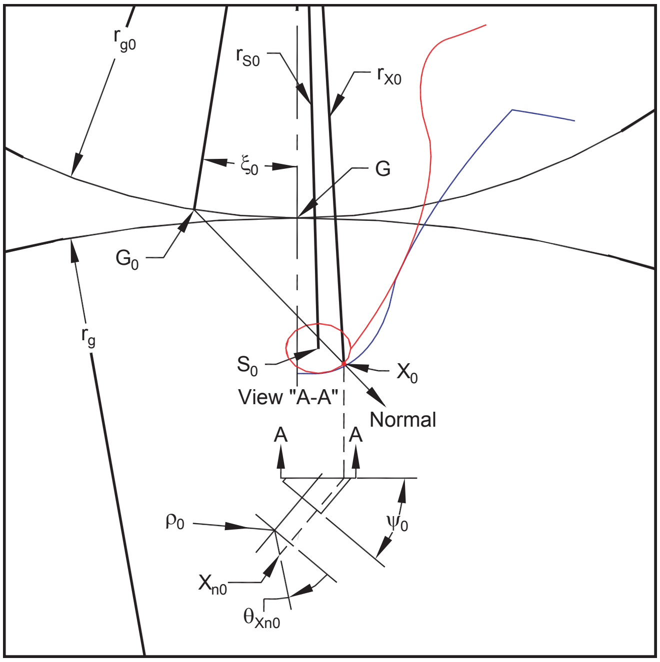

Figure 5—An arbitrary point X_0 and the normal on the tool tip.

The derivation of Equation 44 is given in Appendix A.

Note: For a shaper tool with non-elliptical tool tip, Equations 43 and 44 should be bypassed and the actual tool tip geometry, X_0 and m_{X0}, should be used for the subsequent calculations.

The normal at X_0 can be expressed as a linear equation:

$$y = m_{X0}(x - x_{X0}) + y_{X0} \tag{45}$$

where

- x_{X0} is the x-coordinate of X_0
- y_{X0} is the y-coordinate of X_0

When extended, the normal will intersect the generating pitch circle of the shaper tool at point G_0 (see Fig. 5). The x-coordinate of the intersection point can be calculated as:

$$x_{G0} = \frac{m_{X0}k_2 + \sqrt{r_{g0}^2 k_1 - k_2^2}}{k_1} \tag{46}$$

where

- r_{g0} is the generating pitch radius, tool (in.)
- k_1 is a temporary variable
- k_2 is a temporary variable (in.)

$$k_1 = m_{X0}^2 + 1 \tag{47}$$

$$k_2 = m_{X0}x_{X0} - y_{X0} \tag{48}$$

The angle, ξ_0, formed between the y-axis and the tool radius at the intersection point, G_0, is

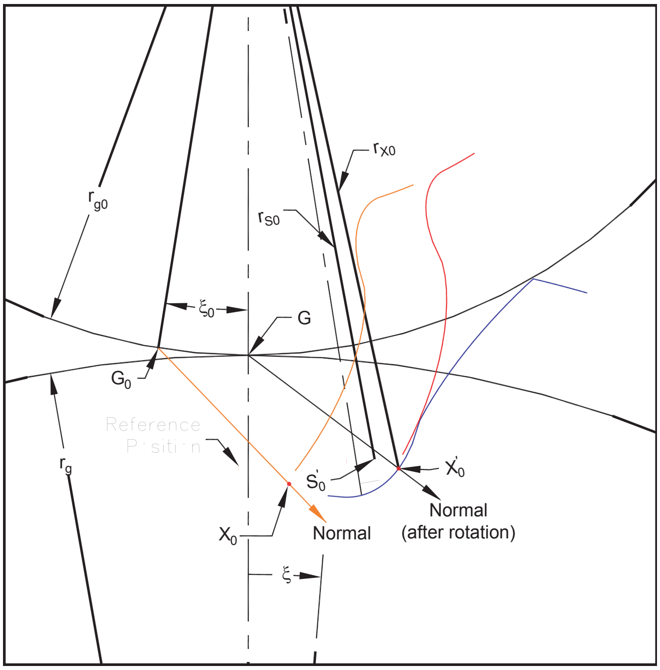

Figure 6—The arbitrary point X_0 and the normal after rotating the shaper tool for an angle $-\xi_0$.

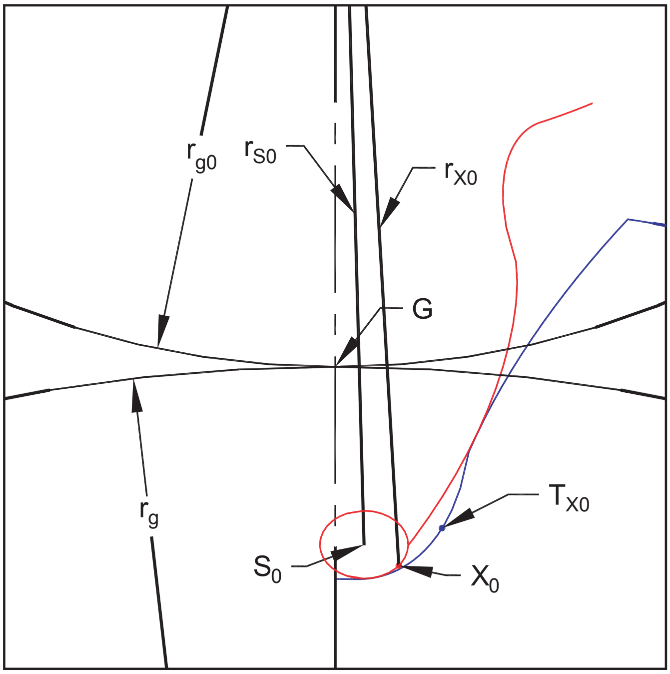

Figure 7—The arbitrary point X_0 on the tool tip and its conjugate point T_{X0} on the trochoid.

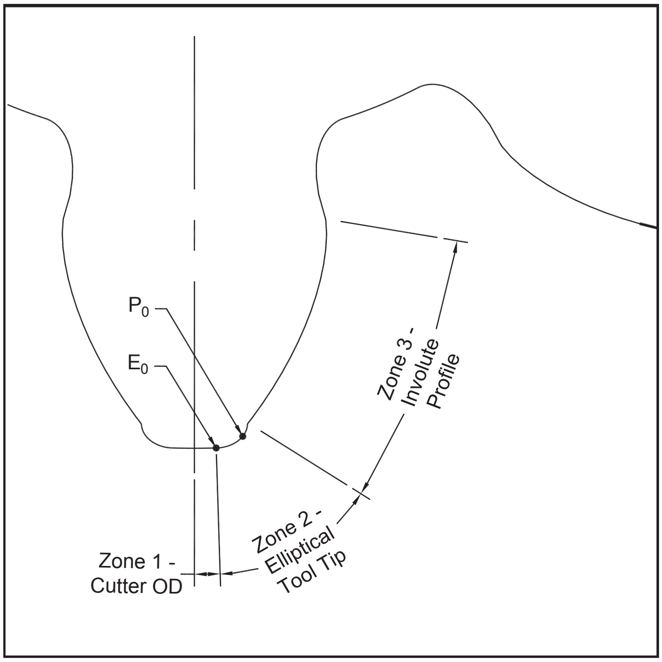

Figure 8—Zones of a shaper cutter.

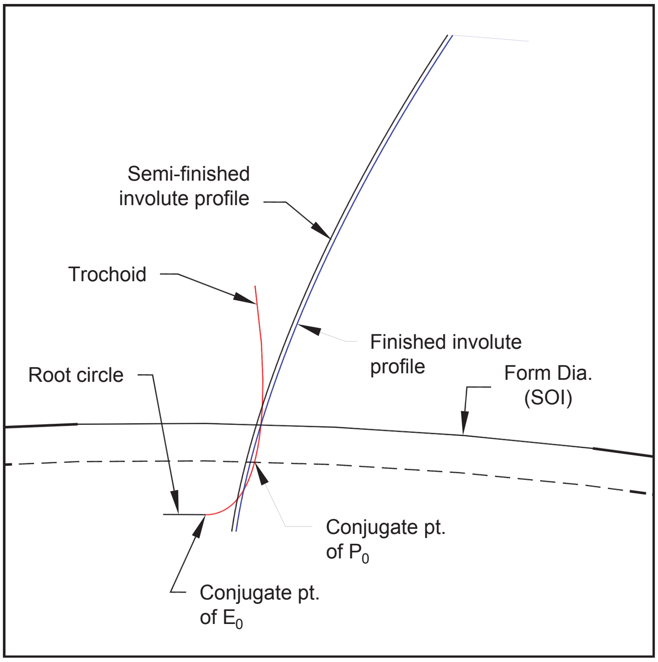

Figure 9—Shaper-tool-generated fillet profile (trochoid).

$$\xi_0 = \arcsin\left(\frac{x_{G0}}{r_{G0}}\right) \tag{49}$$

Note: The angle ξ_0 may be positive or negative. If G_0 is on the left side of the y-axis, x_{G0} (see Eq. 49) will be negative, and so will ξ_0. On the other hand, if G_0 is on the right side of the y-axis, ξ_0 will have a positive value.

To find the conjugate point of X_0, the shaper tool is rotated from its reference position (see Fig. 6) by an angle, $-\xi_0$. The arbitrary point X_0 will rotate to a new position, X_0'

$$X_0' = M(-\xi_0)X_0 \tag{50}$$

where

$$M(\varphi) = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \tag{51}$$

$M(\varphi)$ is a rotation matrix. When multiplied to a vector, the vector would be rotated an angle φ about the origin (0,0). If $\varphi > 0$, the rotation is counterclockwise. Otherwise, the rotation is clockwise.

After rotating the cutter (see Fig. 6), the normal at the arbitrary tool tip point (now X_0') will pass through the generating pitch point G, thus satisfying the *law of conjugate action* (Ref. 1):

To transmit uniform rotary motion from one shaft to another through the action between two geometric surfaces, the normal to the mating profiles, at the point of contact, must always pass through the same point on the common centerline.

It follows that X_0' is a common point on the tool tip and the trochoid of the gear.

Since the shaper tool and the gear rotate in a constant speed ratio, and the shaper tool has rotated an angle, $-\xi_0$, from its reference position, the gear rotation angle would have rotated an angle ξ, where

$$\xi = \pm\, \xi_0 \, \frac{n_0}{n} \tag{52}$$

To return the gear to its reference position, it is rotated an angle $-\xi$ about the gear center O_G. After rotating the gear, the common point X_0' will move to T_{X0} (see Fig. 7), which is a point on the trochoid. T_{X0} can be calculated as:

$$T_{X0} = M(-\xi)(X_0' - O_G) \tag{53}$$

Note: In Equation 53, the origin of T_{X0} (see Eq. 53) is the center of the gear O_G, not the center of the tool.

Determination of a Shaper-Tool-Generated Fillet Profile. The shaper tool discussed in this article can be divided into three zones (see Fig. 8):

1) Zone 1 is the portion of cutter profile that coincides with the outside diameter of the shaper tool. It starts from the outside diameter of the cutter on the y-axis, and ends at the end tangent point, E_0. The tool profile in this zone generates the root circle of the gear. If the shaper tool has a full tip radius, Zone 1 reduces to a single point on the outside diameter of the tool.
2) Zone 2 is the elliptical tool tip starting at E_0 and ends where the tool tip joins the main shaper tool profile (Zone 3).
3) Zone 3 is the main cutter profile that generates the involute profile on the semi-finished gear.

The shaper-tool-generated trochoid can be determined by calculating the conjugate points of the tool tip in Zone 2. Begin the calculation at E_0 (see Fig. 8), and continue in small increments towards P_0. The conjugate point of P_0 will usually penetrate deepest from the surface of the involute profile (see Fig. 9). Continue the calculation procedure until the trochoid intersects the involute tooth profile. Additional trochoid points can be calculated if desired.

Using Shaper Tool Algorithm to Calculate Fillet Profile of a Hobbed Gear

The tooth profile of a shaper tool with an infinite number of teeth will approach a rack. Naturally, if a shaper tool algorithm could handle an infinite number of tool teeth, a hobbed trochoid could be accurately approximated. Unfortunately, the shaper tool algorithm presented in this article does not allow for an infinite number of tool teeth. A shaper tool with a finite, but large number of teeth is permitted.

To investigate the feasibility of approximating a hobbed trochoid with the shaper algorithm using a shaper tool with a large number of teeth, a numerical example was calculated using Example 3.1.5 of AGMA 918-A93 (Ref. 8). The number

Table 1—Comparison of a Hobbed Pinion Fillet Profile (Ex. 3.1.5 - AGMA 918-A93) with Fillet Profiles Generated with 100-; 1,000-; and 10,000-Tooth Shaper Tools.

Description		Gear data	Tool data			
			Hobbed	100T-Shaper	1,000T-Shaper	10,000T-Shaper
Normal diametral pitch	in.$^{-1}$	12	12	12	12	12
Number of teeth		35	NA	100	1,000	10,000
Reference normal pressure angle	deg.	20	20	20	20	20
Reference helix angle	deg.	22.109	22.109	22.109	22.109	22.109
Outside diameter (or hob addendum)	in.	3.3686	0.1205	9.2357	90.1882	899.7129
Reference normal circular thickness	in.	0.1501	0.1309	0.1309	0.1309	0.1309
Stock allowance	in.	0.001	NA	NA	NA	NA
Tool tip radius	in.	NA	0.0100	0.0100	0.0100	0.0100
Protuberance	in.	NA	0.0025	0.0025	0.0025	0.0025
Comparison of the calculated fillet profile						
Maximum difference between hobbed & shaped profiles	in.	NA	NA	0.001901	0.000111	0.000011
Comparison of form diameter (SOI)						
Form diameter	in.	NA	3.040483	3.050692	3.041641	3.040600
Difference between hobbed & shaper-generated form diameters	in.	NA	NA	0.010209	0.001158	0.000117

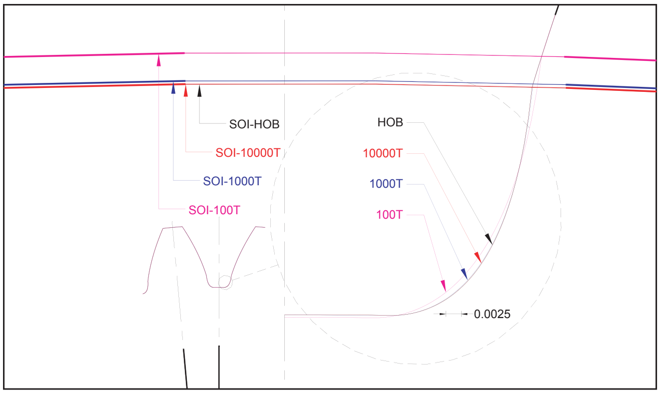

Figure 10—Pinion trochoid (Ex. 3.1.5-AGMA 918-A93) generated with a hob and shaper cutters.

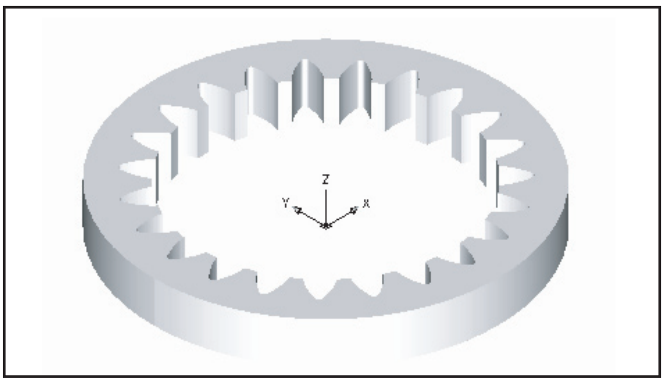

Figure 11—A 23-tooth internal spur gear model.

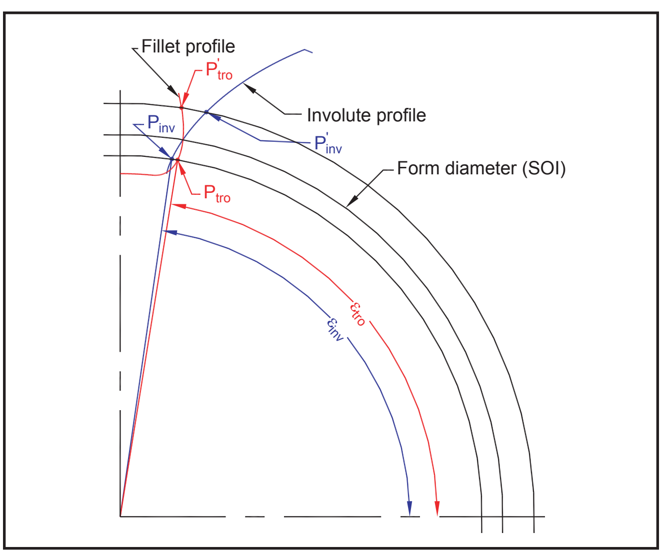

Figure 12—Polar angles of a trochoid point and an involute point.

of shaper tool teeth used were 100, 1,000 and 10,000. Table 1 compares the distances between the trochoid curves generated with the shaper cutters and the one generated with a hob. The form diameters or the start of involute (SOI) (to be discussed in the section "Calculating Form Diameter," below) based on the shaper tool were also compared to that generated with a hob. Figure 10 shows the trochoid curves superimposed on each other for a visual comparison.

Table 1 showed that the maximum distance between the trochoid curves generated with a 100-tooth shaper cutter and the hob to be 0.001901". For a 10,000-tooth shaper cutter, the difference decreased to merely 0.000011". The difference between the shaper-generated and the hobbed SOI's followed a similar trend. For the 100-tooth shaper cutter, the difference was 0.010209" and for 10,000-tooth shaper cutter, 0.000117".

The trochoid curves plotted in Figure 10 show the shaper-generated trochoid converging to that of the hobbed one when the number of teeth in the shaper tool is large (e.g. 10,000).

Applications for the Shaper Tool Algorithm

The shaper tool algorithm can be used in computer-aided gear design and gear tooth modeling as shown in Figure 11. The algorithm is also useful for calculating the trochoid geometry for finite element or boundary element analysis. The following sections describe applications of the shaper tool algorithm in form diameter calculation and gear finishing stock analysis.

Calculating Form Diameter

The form diameter or the start of involute (SOI) of a finished gear is the gear diameter where the trochoid joins or intersects the involute profile. When the two curves intersect, two intersection points may appear to exist. The intersection point that is closer to the tip diameter of the gear is the SOI. The other "intersection" point is an artificial one, as the involute curve has already been truncated at the SOI. When calculating the SOI of a gear by iteration, it is important to make sure that the algorithm converges to the SOI. Plotting the trochoid and the involute profile will provide a visual verification that the iteration process converges correctly (see Fig. 12).

The SOI can be calculated by comparing the polar angles of a trochoid point and an involute profile point, ε_{tro} and ε_{inv} respectively, on the same gear diameter (see Fig. 12) (Ref. 9). When the two polar angles become equal, the trochoid and involute points will coincide, and the gear diameter at the intersection point is the SOI. If the two polar angles are unequal, compare the polar angles for a new set of points at slightly larger or smaller gear diameter than the current one. Repeat the process until the two polar angles become equal.

Table 2 compares the calculated SOI's of the selected numerical examples in AGMA 918-A93 (Ref. 8) using the shaper tool algorithm presented in this article and those using other gear software. For hobbing examples, a 10,000-tooth shaper tool was used for the trochoid calculation. The calculated SOI's using the shaper tool algorithm compared well with those using other software.

Checking Gear Finishing Stock

For gears finished by grinding or shaving, the semi-finishing tool is usually designed with protuberance that would generate an undercut in the gear. The protuberance provides stock for finishing operations. The form diameter (SOI) of the finished gear must be smaller than the start of active profile (SAP) of the gear when the gear meshes with the mate. The algorithm presented in this article can verify if a semi-finishing tool would provide sufficient finishing stock on the gear while keeping the SOI smaller than the SAP.

Consider a helical gear set with the basic geometry given in Table 3. The initial pinion hob (A) design used the same standard reference pressure angle, 20°, as the part. Consequently, the calculated SOI (4.4873") was larger than the SAP (4.4788").

Table 2—Comparison of the Form Diameters Calculated Using the Proposed Algorithm and Other Software.

Description		Example 3-1-1		Example 3-1-3		Example 3-1-9	
Gear data		Pinion	Gear	Pinion	Gear	Pinion	Gear
Gear type		Spur		Single helical		Internal helical	
Normal diametral pitch	in.$^{-1}$	5	5	6	6	9	9
Number of teeth		51	104	21	86	24	69
Ref. norm. press. angle	deg.	20.0000	20.0000	20.0000	20.0000	25.0000	25.0000
Standard helix angle	deg.	0.0000	0.0000	15.0000	15.0000	17.7276	17.7276
Normal circular thickness	in.	0.326267	0.293451	0.322622	0.257794	0.217257	0.192968
Stock allowance	in.	0.008000	0.008000	0.005300	0.005300	0.000000	0.000000
Tool data							
Tool type		Hob	Hob	Hob	Hob	Shaper	Shaper
Number of teeth		10,000	10,000	10,000	10,000	36	36
Addendum/Outside diameter	in.	0.291300	0.291300	0.246000	0.246000	4.295000	4.476600
Normal circular thickness	in.	0.314200	0.314200	0.261800	0.261800	0.102100	0.186000
Tool tip radius	in.	0.067300	0.067300	0.068200	0.068200	0.020000	0.012000
Protuberance	in.	0.009500	0.009500	0.008000	0.008000	0.000000	0.000000
Calculated form diameters							
SOI–based on this paper	in.	9.921823	20.204571	3.489356	14.525332	2.676943	8.225700
SOI–from other software	in.	9.921617	20.204577	3.489576	14.525135	2.676900	8.225700
Difference	in.	0.000206	–0.000006	–0.000220	0.000197	0.000043	0.000000

Table 3—Comparison of Form Diameters of a Pinion Generated with Normal Lead and Short Lead Hobs.

Description	Unit	Oper. Cntr. Dist. 14.500 in.		Pinion Hob A (normal lead)	Pinion Hob B (short lead)
		Gear	Pinion		
Normal diametral pitch	in.$^{-1}$	4.0000		4.0000	4.1211
Number of teeth		93	18	10,000	10,000
Ref. norm. pressure angle (part or hob)	deg.	20		20	14.5
Reference helix angle	deg.	15.1560		15.1560	14.7003
Outside diameter (or hob addendum)	in.	24.5840	5.4160	0.3372	0.1373
Reference normal circular thickness	in.	0.3874	0.4812	0.3889	0.2419
Stock allowance per flank	in.	0.0050	0.0050	NA	NA
Tool tip radius	in.	NA		0.0900	0.0900
Protuberance	in.			0.0070	0.0070
Comparison of SOI and SAP					
Start of active profile (SAP)	in.			4.4788	4.4788
Form diameter (SOI)	in.			4.4873	4.4550
				SOI>SAP	SOI<SAP

Therefore, hob (A) does not provide the required grinding stock while keeping the SOI below the required SAP. In order to push the SOI closer to the root diameter, a short lead hob (B) was designed. The hob had a 14.5° reference normal pressure angle. The calculated SOI based on the short lead hob (B) was 4.4550", smaller than the SAP. Hob B provided the required finishing stock with satisfactory SOI (see Fig. 13).

Conclusions

A method for determining the shaper-tool-generated fillet profile (trochoid) was presented. The method is applicable to both external and internal helical gears. The algorithm is based on a class of shaper tool that has an involute main profile and elliptical tool tip in the transverse plane. However, the algorithm will also work for a shaper tool with other tool tip geometries, provided the coordinates and the normal of the tool tip profile are known.

The shaper tool algorithm can also approximate the trochoid generated with a rack-type tool if the number of shaper tool teeth is large. The numerical examples showed that a trochoid curve generated with a 10,000-tooth shaper tool can approximate that generated with a hob with small error.

The algorithm presented in this article does not require the

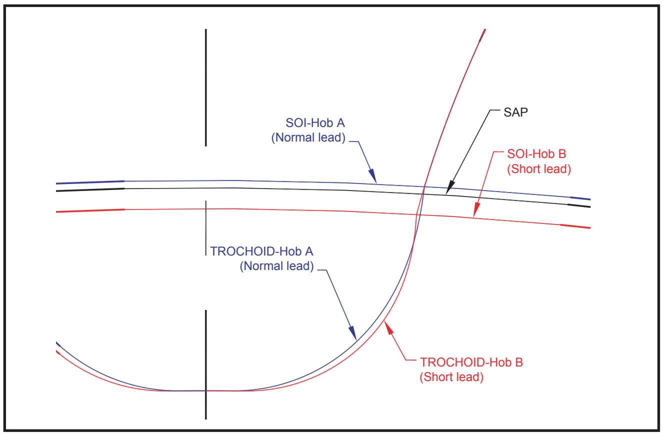

Figure 13—Trochoid curves generated with a normal lead and a short-lead hob.

tool and the gear to have equal reference normal pressure angle. Consequently, a trochoid generated with a non-standard cutter such as a short lead hob can also be calculated.

Examples for the form diameter (SOI) calculation and the finishing stock analysis were provided using the shaper tool algorithm presented.

A computer program was developed using the algorithm described in this article. The calculated form diameters (SOI's) for both external and internal gears compare well to those calculated with other gear software. An internal spur gear was used to verify the shaper tool algorithm.

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Appendix A—Derivation of Equations

The tangent and the normal of an arbitrary point on the shaper tool tip (Equations 20 and 44). The shaper tool tip considered in this article is circular in the normal plane and elliptical in the transverse plane, as shown in Figure A1. (Ref. A1). The coordinates of an arbitrary tool tip point X_0 (related to the center of the tool tip) in the transverse plane can be calculated as

$$x_{X0} = \rho_0 \frac{\cos\theta_{Xn0}}{\cos\psi_0} \quad (A.1)$$

$$y_{X0} = \rho_0 \sin\theta_{Xn0} \quad (A.2)$$

where

ρ_0 is the tool tip radius and
θ_{Xn0} is the auxiliary angle for point X_0 measured clockwise from the horizontal axis.

Differentiating Equation A.1 and Equation A.2 with respect to the auxiliary angle θ_{Xn0} we get

$$dx_{X0} = -\rho_0 \frac{\sin\theta_{Xn0}}{\cos\psi_0} d\theta_{Xn0} \quad (A.3)$$

$$dy_{X0} = \rho_0 \cos\theta_{Xn0}\, d\theta_{Xn0} \quad (A.4)$$

The slope of the tangent at point X_0 can be calculated as

$$\tan\alpha_{X0} = \frac{dy_{X0}}{dx_{X0}} = -\frac{\cos\psi_0}{\tan\theta_{Xn0}} \quad (A.5)$$

Similarly, the slope of the profile tangent point, P_0 (see the section "Center of tool tip on a shaper tool," above), can be calculated as

$$\tan\alpha_{P0} = -\frac{\cos\psi_0}{\tan\theta_{Pn0}} \quad (A.6)$$

Taking an arc tangent on both sides of Equation A.6 completes the derivation for Equation 20.

$$\alpha_{P0} = \arctan\left(\frac{-\cos\psi_0}{\tan\theta_{Pn0}}\right) \quad (A.7)$$

The normal at the given arbitrary point on a shaper tool tip is perpendicular to the tangent. Therefore, the slope of the normal, m_{X0} (see Eq. 44), is:

$$m_{X0} = \frac{-1.0}{\tan\alpha_{X0}} = \frac{\tan\theta_{Xn0}}{\cos\psi_0} \quad (A.8)$$

Generating pressure angle (Equation 39). The generating pressure angle is based on tight meshing of a shaper tool with a semi-finished gear. The derivation of the generating pressure angle equation is similar to the one given in 86 FTM 1 (Ref. A2).

The following tool and gear data are given:

s_{b0} is the transverse base circular thickness, tool (in.);
r_{b0} is the base radius, tool (in.);
s_{br} is the transverse base circular thickness, semi-finished gear (in.). If shaping is the finishing operation, the base circular thickness for the finished gear should be used; and
r_{br} is the base radius, semi-finished gear (in.).

The sum of the transverse circular thickness of the tool and the gear equals the circular pitch at the generating pitch circle.

$$p_{g0} = s_{g0} + s_{gr} \quad (A.9)$$

where
p_{g0} is the transverse circular pitch at the generating pitch circle;
s_{g0} is the transverse circular thickness at the generating pitch circle, tool (in.); and
s_{gr} is the transverse circular thickness at the generating pitch circle, gear (semi-finished) (in.).

The circular thicknesses of tool and semi-finished gear at the generating pitch circle can be calculated as

$$s_{g0} = 2r_{g0}\left(\frac{s_{b0}}{2r_{b0}} - \mathrm{inv}\phi_g\right) \quad (A.10)$$

$$s_{gr} = 2r_{gr}\left(\frac{s_{br}}{2r_{br}} \mp \mathrm{inv}\phi_g\right) \quad (A.11)$$

where
r_{g0} is the generating pitch radius of the shaper tool (in.);
r_{gr} is the generating pitch radius of the semi-finished gear (in.); and
inv ϕ_g is the involute function of the generating pressure angle, ϕ_g.

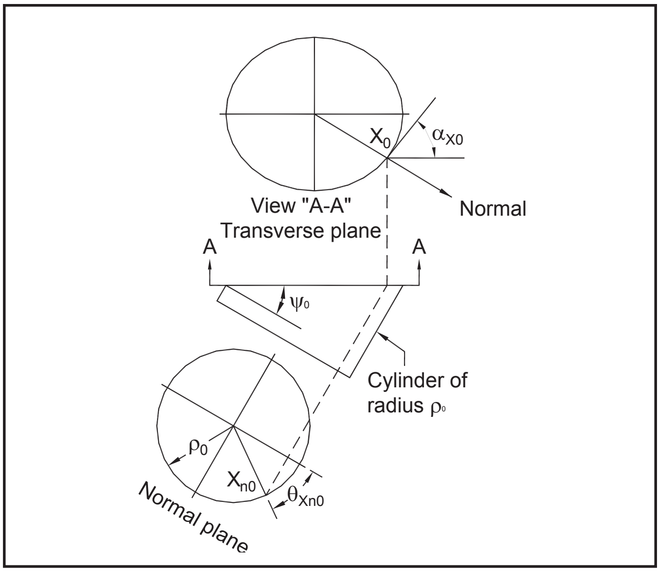

Figure A1—Shaper tool tip in normal and transverse planes.

Substituting Equation A.9 and Equation A.10 into Equation A.8 and dividing both sides of the new equation by $2r_{g0}$, we get

$$\frac{p_{g0}}{2r_{g0}} = \frac{s_{b0}}{2r_{b0}} - \mathrm{inv}\phi_g + \frac{r_{gr}}{r_{g0}}\frac{s_{br}}{2r_{br}} \mp \frac{r_{gr}}{r_{g0}}\mathrm{inv}\phi_g \quad (A.12)$$

using the following established relationships

$$\frac{p_{g0}}{2r_{g0}} = \frac{p_{b0}}{2r_{b0}} \quad (A.13)$$

$$\frac{r_{gr}}{r_{g0}} = \frac{r_{br}}{r_{b0}} \quad (A.14)$$

where

p_{b0} is the transverse base circular pitch, tool (in.).

Substituting Equation A.12 and Equation A.13 into Equation A.11, we get

$$\frac{p_{b0}}{2r_{b0}} = \frac{s_{b0}}{2r_{b0}} - \mathrm{inv}\phi_g + \frac{r_{br}}{r_{b0}}\frac{s_{br}}{2r_{br}} \mp \frac{r_{br}}{r_{b0}}\mathrm{inv}\phi_g \quad (A.15)$$

Multiply both sides of Equation A.14, by $2r_{b0}$ and solve for inv ϕ_g (Eq. 39)

$$\mathrm{inv}\phi_g = \frac{s_{b0} + s_{br} - p_{b0}}{2(r_{b0} \pm r_{br})} \quad (A.16)$$

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