Pressure Angle Changes in the Transverse Plane for Circular Cut Spiral Bevel Gears

by
Ronald Huston
University of Cincinnati
Cincinnati, Ohio

and

John J. Coy
NASA Lewis Research Center
Cleveland, Ohio

Abstract

This article examines pressure angle changes along a tooth for circular cut spiral bevel crown gears. The changes are measured in the transverse planes for various cutter profiles. Three cases are considered: 1) a straight line profile; 2) a circular profile; and 3) an involute profile. In each case, the heel-to-toe variation is approximately 3, depending on the cutter radius. For conical gears, the variation is increased by the factor $1/\sin\alpha$ where α is the half-cone angle. Finally, it is shown that pressure angle variation occurs for all cutter profiles.

Introduction

Recently it has been suggested that the transverse plane may be very useful in studying the kinematics and dynamics of spiral bevel gears.[1,2] The transverse plane is perpendicular to the pitch and axial planes as shown in Fig. 1. Buckingham[3] has suggested that a spiral bevel gear may be viewed as a limiting form of a "stepped" straight-toothed gear as in Fig. 2. The transverse plane is customarily used in the study of straight toothed bevel gears.[4]

For spiral bevel gears the *normal* plane is often used for studying the kinematics and dynamics. One reason for this is that for smooth tooth surfaces the contact forces between mating teeth are transmitted in the normal plane; that is, the resultant force vector is in the normal plane. However, if friction is present, the resultant force vector is rotated out of the normal plane, and it becomes more nearly parallel to the transverse plane.

Therefore, in this brief article the pressure angle changes in the transverse planes are examined along the circular cut tooth. The balance of the article contains four parts. The first part provides some preliminary geometrical considerations. The next part contains the analysis. Applications are presented in the third part, and the final part makes some conclusions for mechanical design.

AUTHORS:

DR. RONALD L. HUSTON *is presently employed as a Professor of Mechanics in the Department of Mechanical and Industrial Engineering for the University of Cincinnati. He also serves as the Director of the University's Institute for Applied Interdisciplinary Research. Dr. Huston was awarded a Ph.D. in Engineering Mechanics, an M.S. in Engineering Mechanics, and a B.S. in Mechanical Engineering from the University of Pennsylvania.*

DR. JOHN J. COY *is employed by the U.S. Army Aviation Systems Command's Propulsion Directorate at the NASA Lewis Research Center in Cleveland. Presently, he serves as the Head, Rotercraft Section in the Aeronautics Directorate. He earned Ph.D., M.S. and B.S. Degrees in Mechanical Engineering through the University of Cincinnati. He is Chairman of the ASME Power Transmission and Gearing Committee and holds membership on the NATO/AGARD Propulsion Energetics Panel.*

Preliminary Geometrical Concepts

Consider the pitch plane and a circular-cut tooth centerline as shown in Fig. 3. X and Y are the Cartesian axes with origin at 0, the gear center. X_1 and Y_1 are axes parallel to X and Y with origin at C, the cutter center. C has X and Y coordinates (H, V). R_c is the mean "mean cutter radius"; that is, R_c is the distance from C to the tooth surface in the pitch plane. The cutter radius r_1 for other points on the tooth surface is a function of the elevation z of those points above or below the pitch plane. For example, for and "inside" tooth surface r_1 might be expressed as:

$$r_1 = R_c + F(z) \tag{1}$$

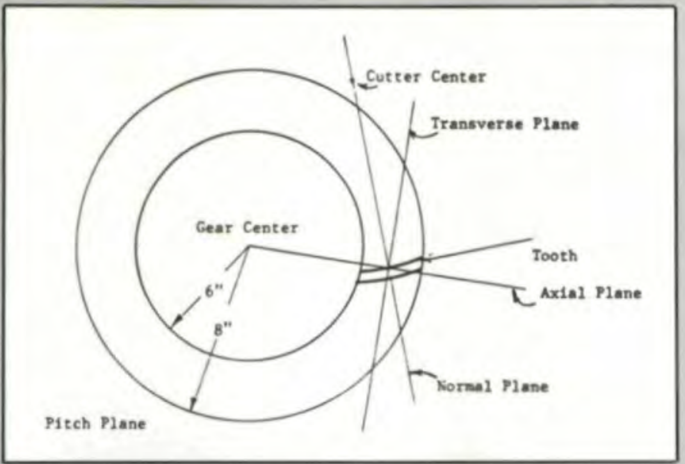


Fig. 1 – Spiral Bevel Crown Gear and Identifying Planes

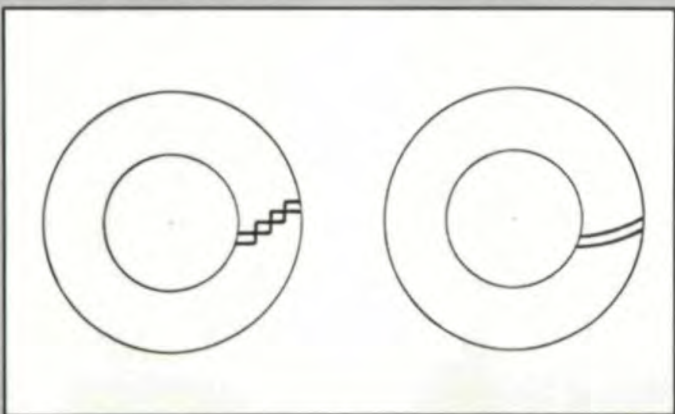


Fig. 2 – Spiral Bevel Gear as a Limiting Form of a "Stepped" Straight-Toothed Gear

where F(z) describes the cutter tooth geometry. Finally in Fig. 3, R_i and R_o are the inner ("toe") and outer ("heel") tooth radii and ψ_m is the spiral angle at the mean tooth radius R_m.

Consider a typical point P along the tooth centerline. Let r be the radial distance OP and let ϵ be the angle between OP and the X axis as shown in Fig. 4. Let X_2 and Y_2 be a third coordinate system and let its origin be C and let it be inclined at an angle ϵ to X_1 and Y_1. Then X_2 is parallel to OP. Finally, let ψ be the spiral angle at P as shown.

The angle ψ and the radial distance r are not independent, but are related by the expression[2,4]

$$\sin\psi = (r_2 + R_c^2 - H^2 - V^2)/2rR_c \tag{2}$$

(This relation follows from the law of cosines with triangle OPC by noting that the cosine of the angle at P is $\sin\psi$.) If R_m is the mean tooth radius, Equation (2) may be expressed as:

$$\sin\psi = (r_2 - R_m^2 + 2R_mR_c \sin\psi_m)/2rR_c \tag{3}$$

(This relation is obtained by noting in Fig. 3 that H and V may be expressed as $H - R_m - R_c\sin\psi_m$ and $V = R_c \cos\psi_m$.)

The tooth profile in the transverse plane at the typical point

Nomenclature

- α – Half-Cone Angle
- ϵ – Angle Between OP and the X-axis (Fig. 4)
- ψ – Spiral Angle
- ψ_m – Spiral Angle at the mean Tooth Radius
- ϱ – Radius of the Circular Profile and Involute Generating Circle
- θ – Pressure Angle in the Transverse Plane
- θ_c – Complement of the Pressure Angle
- β – Pressure Angle in the Normal Plane
- ξ, η – Coordinate System in the Normal Plane (Fig. 8)
- a,b – Coordinates of the Center of a Circular Tooth Profile (Fig. 7)
- C – Cutter Center
- F(z) – Function Defining the Cutter Tooth Geometry
- H, V – X and Y Coordinates of C
- 0 – Gear Center
- r – Radial Distance
- r_1 – General Cutter Radius
- R_c – Mean Cutter Radius
- R_i – Inner ("Toe") Tooth Radius
- R_o – Outer ("Heel") Tooth Radius
- R_m – Mean Tooth Radius
- X, Y – Cartesian Axes with Origin at 0
- X_1, Y_1 – Cartesian Axes with Origin at C
- X_2, Y_2 – Cartesian Axes with Origin at C (Fig. 4)
- z – Elevation Above or Below the Pitch Plane

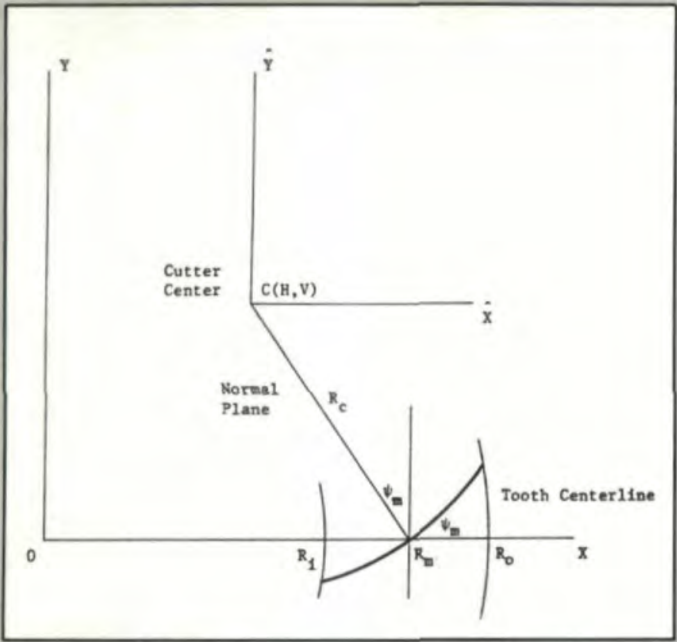


Fig. 3 – View of Pitch Plane and Tooth Centerline of Crown Gear

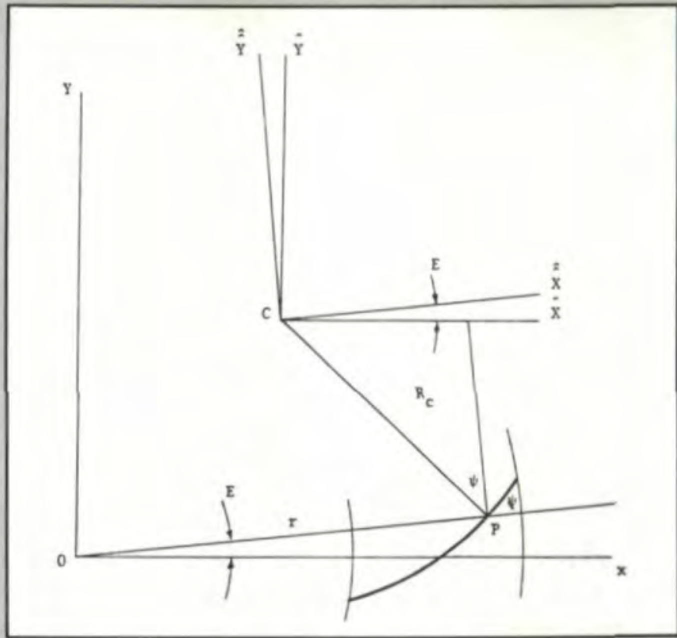


Fig. 4 – Coordinate Geometry for a Typical Point P on the Tooth Centerline

P depends upon r (or ψ). The following part of the article containing the central analysis, discusses these tooth profile changes between different transverse planes along the tooth centerline for various cutter profiles.

Analysis of Tooth Profile Changes Between Transverse Planes

Equation (1) provides a relationship between the cutter radius r_1 and the elevation z of a point on the tooth surface. By solving for z, the relationship may be expressed in the form:

$$z = f(r_1) \tag{4}$$

The cutter profile, described by the function F(z) of Equation (1), is thus also described by the function $f(r_1)$ in Equation (4); however, in Equation (4), the ensuing tooth surface is readily seen to be a surface of revolution. Equation (4) may also be viewed as providing a description of the tooth profile in the normal plane.

The cutter radius r_1 may be expressed in terms of the coordinates x_1, y_1 and x_2, y_2 in the form:

$$r_1 = (x_1^2 + y_1^2)^{1/2} = (x_1^2 + y_2^2)^{1/2} \tag{5}$$

Hence, by comparing Equations (4) and (5), z may be considered to be a function of x_1 and y_1 or of x_2 and y_2. If x_1 or x_2 has a constant value, Equation (4) provides a description of the tooth profile in a *transverse* plane. For example, if $x_1 = R_c\sin\psi_m$, the tooth profile in the mid-transverse plane is

$$z = f\,([R_c^2\sin^2\psi m + y_1^2]^{1/2}) = g(y_1, \psi_m) \tag{6}$$

That is, the elevation of a point on the tooth surface in the mid-transverse plane depends upon y_1. For a general transverse plane, Equation (6) may be expressed as:

$$z = f\,([R_c^2\sin^2\psi + y_2^2]^{1/2}) = g(y_2, \psi) \tag{7}$$

Thus, the tooth profile in a general transverse plane depends upon the spiral angle ψ which, in turn, is a function of the radial distance r, through Equation (3).

Equation (7) can be used to study tooth profile changes between the transverse planes. For example, a comparison of $g(y_2, \psi)$ with $g(y_2, \psi_m)$ provides a measure of the modification of the transverse profile from the profile in the mid-transverse plane. Equation (7) is also useful for determining the pressure angle changes between the transverse planes. To see this, consider the profile in the transverse plane depicted in Fig. 5. Let θ be the pressure angle and let θ_c be its complement. Then, for the inside tooth surface $\tan\theta_c$ is:

$$\tan\theta_c = \partial z/\partial(y_2) = -(dz/dr_1)(\partial r_1/\partial y_2) \tag{8}$$

or

$$\tan\theta_c = -f'(r_1)\, y_2/r_1 \tag{9}$$

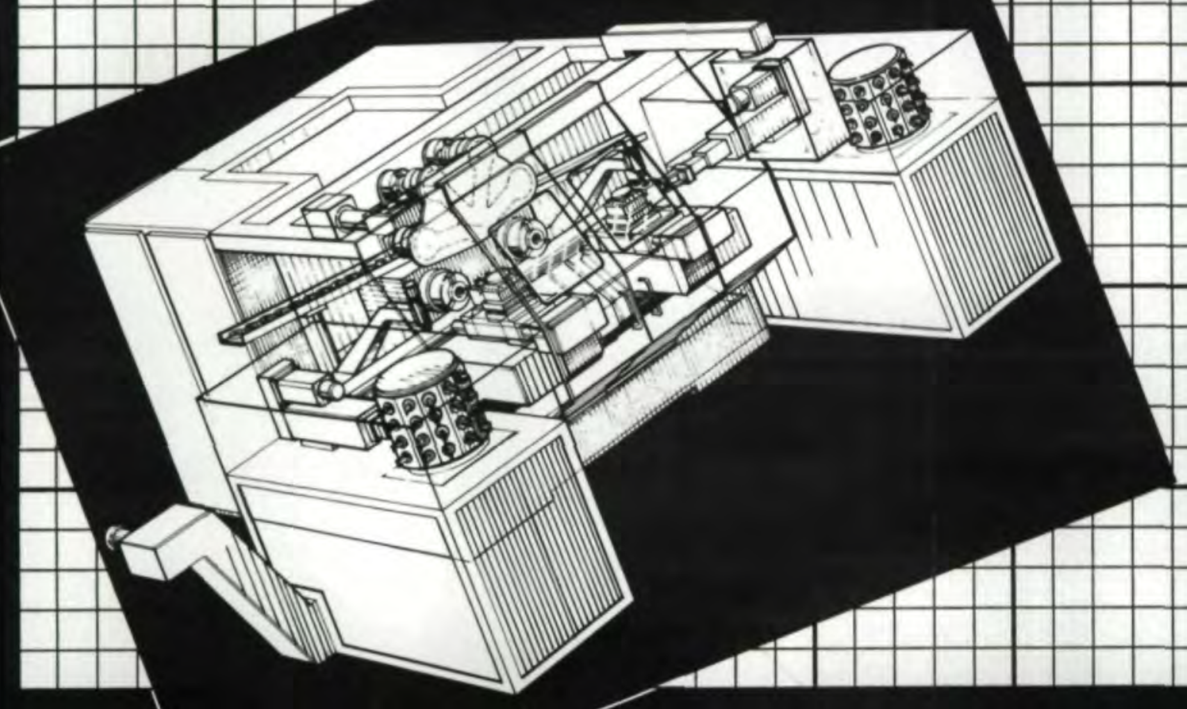
But since $\tan\theta_c = \cot\theta$, the pressure angle θ (in the transverse plane is

$$\theta = -\tan^{-1}[r_1/f'(r_1)y_2] \tag{10}$$

Equation (10) may be viewed as an algorithm which provides the pressure angle as a function of the radial distance r from the gear center. Moreover, it is a valid algorithm for any cutter profile.

Example Applications

Equation (10) was used to study the pressure angle changes



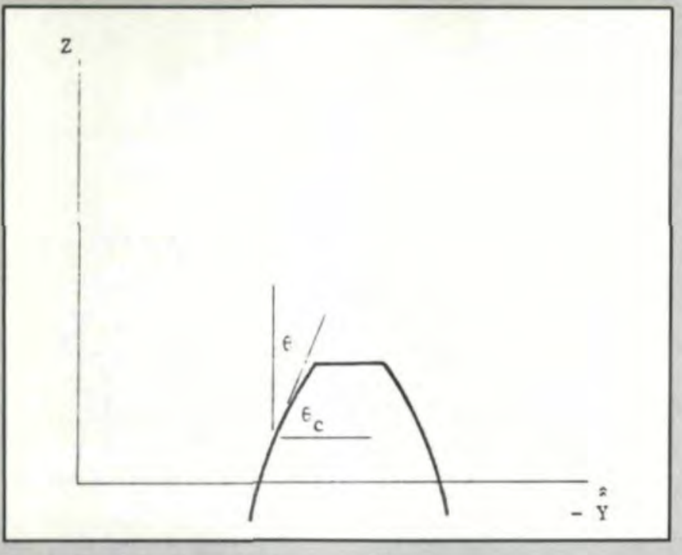


Fig. 5—Tooth Profile in the Transverse Plane

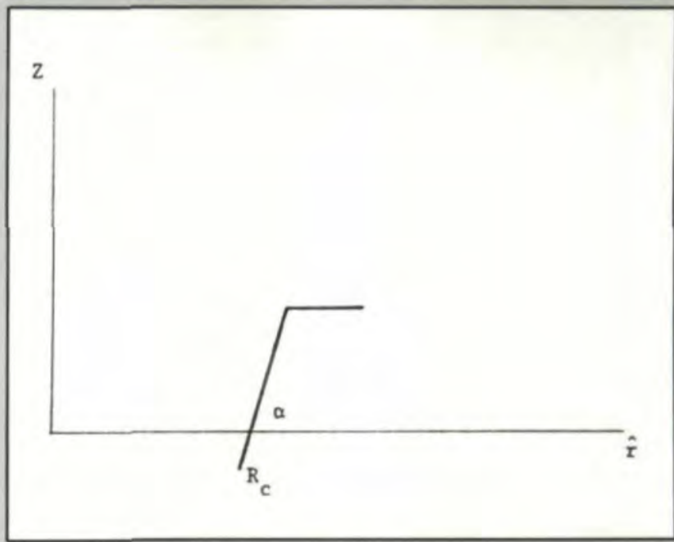


Fig. 6—Straight-Line Tooth Profile in the Normal Plane

through the transverse plane along the inside tooth surface for three cutter profile shapes: 1) a straight line profile, 2) a circular profile, and 3) an involute profile.

1. *Straight Line Cutter Profile.* Fig. 6 depicts a straight line tooth profile in the normal plane. In this case Equation (4) takes the form:

$$z = f(r_1) = (\tan\alpha)(r_1 - R_c) \qquad (11)$$

where R_c is the mean cutter radius and α is the cutter inclination. By substituting into Equation (10) we obtain the transverse plane pressure angle,

$$\theta = \tan^{-1}\{(\cot\alpha)\,[1 + (R_c/y_2)^2 \sin^2\psi]^{1/2}\} \qquad (12)$$

where we have replaced r_1 by $(R_c^2 \sin^2\psi + y_2^2)^{1/2}$ as in Equation (6). The spiral angle ψ may be expressed in terms of the radial distance r by either Equation (2) or (3). Hence, θ is a function of r.

Fig. 9 shows a computer drawn graph of θ in the pitch plane (that is, with $y_2 = -R_c\cos\psi$) for R_c + 6.0 in. (15.24cm), R_m = 7.0 in. (17.78cm), ψm =70°, and α =70.

2. *Circular Cutter Profile.* Fig. 7 depicts a circular tooth profile in the normal plane. In this case, the equation of the profile may be expressed as:

$$(z - b)^2 + (r_1 - a)^2 = \varrho_2 \qquad (13)$$

where a, b, and ϱ are the circle center coordinates and the circle radius as shown in Fig. 7. If α is the cutter inclination at the mean cutter radius, then a and b may be expressed as:

$$a = R_c + \varrho \sin\alpha \text{ and } b = -\varrho \cos\alpha \qquad (14)$$

Hence, Equation (4) may be expressed in the form:

$$z = f(r_1) = -\varrho \cos\alpha + [\varrho_2 - (r_1 - R_c - \varrho \sin\alpha)^2]^{1/2} \qquad (15)$$

Then by substituting into Equation (10), we obtain the transverse plane pressure angle:

$$\theta = \tan^{-1}\Big[\{(\varrho^2 - [(R_c^2 \sin^2\psi + y_2^2)^{1/2} - R_c - \varrho \sin\alpha]^2\}.$$
$$(R_c^2 \sin^2\psi + y_2^2)^{1/2}.$$
$$[(R_c^2 \sin^2\psi + y_2^2)^{1/2} - R_c - \varrho \sin\alpha]^{-1}\, y_2\Big] \qquad (16)$$

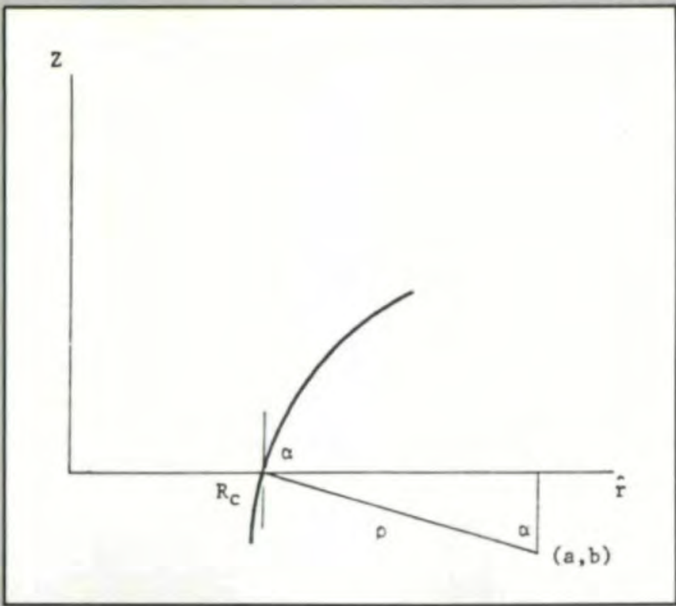


Fig. 7—Circular Tooth Profile in the Normal Plane

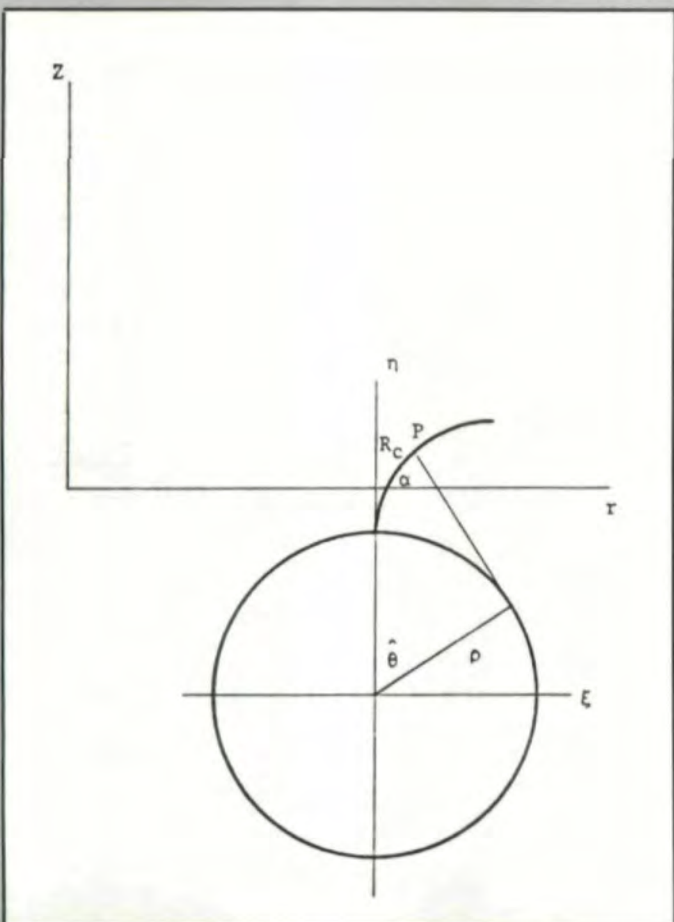


Fig. 8—Involute Tooth Profile in the Normal Plane Together with the Involute Generating Circle

where, as before, we have replaced r_1 by $(R_c^2\sin^2\psi + y_2^2)^{1/2}$.

Fig. 9 also shows a graph of Equation (16) for $R_c = 6.0$ in. (15.24 cm), $R_m = 7.0$ in. (17.78cm), $\varrho = 1.0$ in. (2.54cm), $\psi_m = 30°$, $\alpha = 70°$, and $y_2 = -R_c\cos\psi$.

3. *Involute Profile.* Fig. 8 depicts an involute tooth profile in the normal plane, together with the generating circle of the involute. In terms of the ξ, μ coordinate system, the coordinates of a typical point P on the involute curve may be expressed as:

$$\xi = \varrho\,(\sin\beta - \beta\cos\beta) \tag{17}$$

and

$$\eta = \varrho\,(\cos\beta + \beta\sin\beta) \tag{18}$$

where ϱ is the radius of the generating circle and β is the pressure angle in the normal plane. Equations (17) and (18) are parametric equations of the profile with β being the parameter. In the z, r_1 coordinate system these equations may be written as:

$$z = -\eta_0 + \varrho\,(\cos\beta + \beta\sin\beta) \tag{19}$$

and

$$r_1 = R_c - \xi_0 + \varrho\,(\sin\beta - \beta\cos\beta) \tag{20}$$

where ξ_0 and η_0 are the values of ξ and η when $\beta = (\pi/2) - \alpha$ (that is, ξ_0 and η_0 are the coordinates of the intersection of the profile and the r_1-axis.)

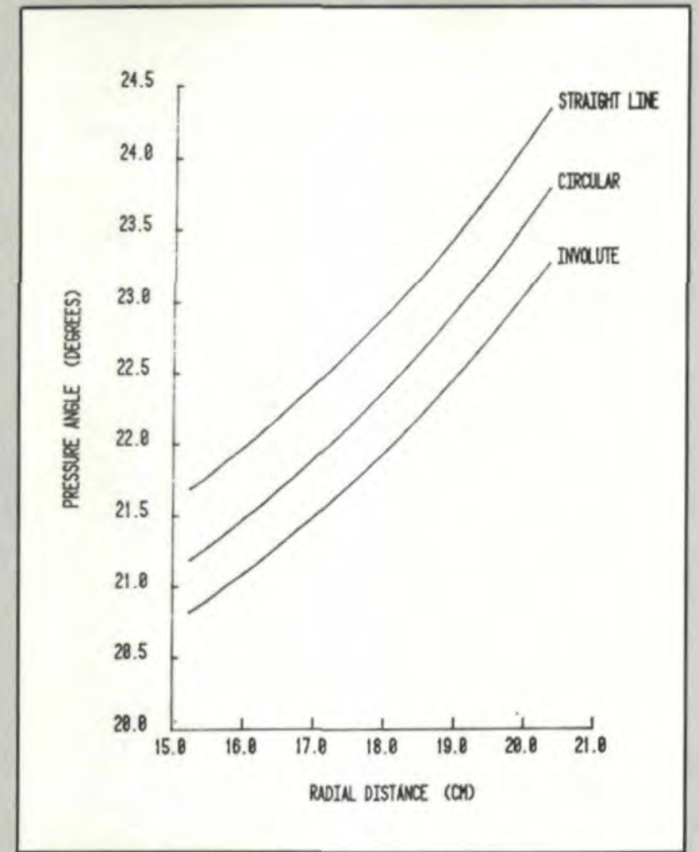


Fig. 9—Pressure Angle Variation for Three Cutter Profiles

In this case, the parametric Equations (19) and (20) replace Equation (4). In Equation (10), $f'(r_1)$ becomes

$$f'(r_1) = dz/dr_1 = (dz/d\beta)/(dr_1/d\beta) = \cot\beta \qquad (21)$$

Hence, the pressure angle in the transverse plane is:

$$\theta = \tan^{-1}[\tan\beta\,(R_c^2\sin^2\psi + y_2)^{1/2}y_2] \qquad (22)$$

where, as before, we have replaced r_1 by $(R_c^2\sin^2\psi + y_2)^{1/2}$, and where in this case, y_2 is related to β through Equation (20), leading to the expression:

$$y_2 = -\{[R_c - \xi_0 + \varrho(\sin\beta - \beta\cos\beta)]^2 - R_c\sin^2\psi\}^{1/2} \qquad (23)$$

Finally, Fig. 9 also shows a graph of Equation (22) for R_c = 6.0 in. (15.24cm), R_m = 7.0 in. (17.78cm) ϱ = 7.0 in. (17.78cm), ψ_m = 30, β = 20°, and y_2 given by Equation (23).

Discussion and Conclusions

Fig. 9 shows the pressure angle variation in the transverse planes for the different cutter profile shapes. In each case the variation is similar, resulting in a pressure angle change of approximately 3° or 15% from heel to toe. For conical gears, this change in pressure angle would be enhanced by the factor $(1/\sin\alpha)$ where α is the half-cone angle.[6]

The effects of this pressure angle change on the gear kinematics, stress, and wear are unknown, but they could be significant.

Finally, the question arises as to whether it would be possible to adjust the cutter profile $f(r_1)$ so that the transverse plane pressure angle would be independent of r, the radial position on the gear. An examination of Equation (10) shows that f is not an explicit function of x_2 or y_2. This means it is *not* possible to adjust f to make $r_1/f'(r_1)y_2$ a constant. Therefore, the pressure angle changes exhibited in Fig. 9 will be similar for all circular cut gears regardless of the cutter profile.

References

1. HUSTON, R. L., and COY, J. J., "Surface Geometry of Circular Cut Spiral Bevel Gears," *Journal of Mechanical Design*, Vol. 104., 1982, pp. 743-748.
2. HUSTON, R. L., LIN, Y., and COY, J. J., "Tooth Profile Analysis of Circular Cut Spiral Bevel Gears," ASME Paper No. 82-Det-79, presented at the ASME Design and Production Engineering Technical Conferences, Arlington, VA 1982.
3. BUCKINGHAM, E., *Analytical Mechanics of Gears*, Dover, New York, 1963, p. 338.
4. BAXTER, M. L., "Basic Theory of Gear-Tooth Action and Generation," *Gear Handbook* (D. W. Dudley, Ed.), McGraw Hill, 1962, Chapter 1.
5. HUSTON, R. L. and COY J. J. "A New Approach to Surface Geometry of Spiral Bevel Gears – Part I: Ideal Gears," *Journal of Mechanical Design*, Vol. 103, No. 4, 1981, pp. 126-132.