# Structural Analysis of Asymmetrical Teeth: Reduction of Size and Weight

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**Abstract**

The present article contains a preliminary description of studies carried out by the authors with a view toward developing asymmetrical gear teeth. Then a comparison between numerous symmetrical and asymmetrical tooth stress fields under the same modular conditions follows. This leads to the formulation of a rule for similar modules governing variations of stress fields, depending on the pressure angle of the non-active side. Finally a procedure allowing for calculations of percentage reductions of asymmetrical tooth modules with respect to corresponding symmetrical teeth, maximum ideal stress being equal, is proposed. Then the consequent reductions in size and weight of asymmetrical teeth are assessed.


**Introduction**

In a paper read at the 24th Annual Italian Association for Stress Analysis (AIAS) Convention (Ref. 1), a geometrical formula for a new type of tooth was presented; this formula is characterised by the fact that the two sides of the tooth have different forms.

The teeth proposed have an asymmetrical form. The two sides of each tooth are characterized by profiles with different pressure angles. These teeth prove useful in mechanisms where the forces employed during rotation in one direction are greater than those engaged in rotation in the opposite one.

The difference of form is obtained by adopting different reference pressure angles ($\alpha_{01} \neq \alpha_{02}$) for the two sides of a tooth.

At a later stage (Ref. 2), in order to compare this new form with the traditional one, a structural analysis using finite elements was carried out; in this study, given the complexity of asymmetrical profiles, it became necessary to devise a specific and totally automatic finite element mesh program. This permitted numerous structural checks covering a broad spectrum of cases and a heterogeneous selection of tooth contours.

All other factors being equal and comparing the results for teeth having different $\alpha_{02}$ values, it appears evident that structural strength increases with the increase of the aforementioned angles.

This implies that in the presence of equal maximum ideal stresses at the root of the teeth, it is possible to create asymmetrically toothed wheels of smaller dimensions with a decrease in weight and bulk, not only of the gear wheels themselves, but also of their boxes and housings.

**SYMBOLS**

The following symbols, based on those recommended by ISO/R 701 (UNI 6773) for notations pertaining to gears, shall be used in the present paper:

| | |
|---|---|
| $m_0$ | reference module |
| $m_{0a}$ | reference module in asymmetrical tooth |
| $G_a$ | weight of asymmetrical tooth |
| $G_0$ | weight of symmetrical tooth |
| $\alpha_{01}$, $\alpha_{02}$ | reference pressure angle of the grip side, during rotation in preferential and non-preferential directions. |
| $\sigma_{i\,max}$ | maximum ideal stress in symmetrical tooth |
| $\sigma_{imaxa}$ | maximum ideal stress in asymmetrical tooth |
| $\Delta\alpha_0$ | $\alpha_{01} - \alpha_{02}$ |

In Ref. 2 variations of stress fields regarding $\alpha_{02}$ were calculated. The maximum ideal stress, $\sigma_{imax}$, corresponding to the mesh's uppermost element, was taken into account; then the value for $\sigma_{imax}$ under the same load and using the same module was compared.

**The Law of Stress Field Variation for Similar Modules**

Keeping in mind the conclusions of the aforementioned studies, the present work aimed at carrying out automated structural analyses of numerous asymmetrical tooth profiles. Then a comparison between these and corresponding symmetrical teeth, having similar modules, was carried out.

The outcome of this thorough investigation confirmed the hypothesis that increases in $\alpha_{02}$ led to a diminution in $\sigma_{imax}$ values; inversely, diminishing $\alpha_{02}$ led to increases in $\sigma_{imax}$ values.

The diminution of $\sigma_{imax}$, with $\alpha_{02} > \alpha_{01}$ is essentially due to an increase in the section close to the internal root contact area. On the basis of this, it is interesting to seek a generally applicable mathematical relationship between $\Delta\sigma_{imax}$ % and $\Delta\alpha_0$, where $\Delta\sigma_{imax}\% = (\sigma_{imaxa} - \sigma_{imax})100/\sigma_{imax}$.

From the results obtained by carrying out numerous comparisons, it emerged that highly accurate variations of $\Delta\sigma_{imax}$ % related to $\Delta\alpha_0$ are obtained by applying the following formula:

$$\Delta\sigma_{imax}\% = \psi\Delta\alpha_0 - \zeta\Delta\alpha_0^2 \quad (1)$$

where $\psi$ and $\zeta$ are two coefficients obtained by interpolating the numerous results obtained for external spur gears.

The relationship described in Eq. 1 is general; it permits us, without carrying

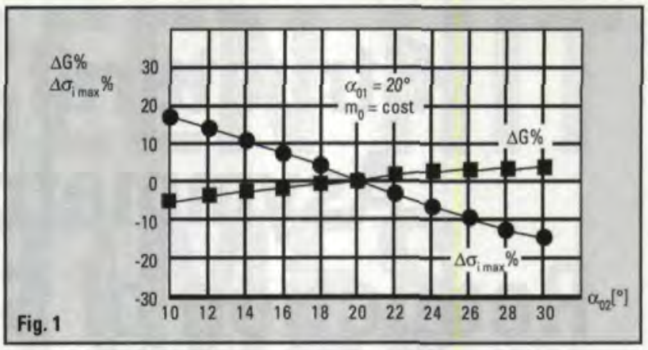


Fig. 1

out structural analyses, to establish readily stress field variations for every kind of asymmetrical tooth in relation to a corresponding symmetrical one.

**Establishment of Reduced Modules in the Presence of Identical Maximum Ideal Stress**

On the basis of the results thus obtained, it has been noted that, in the presence of identical modules, the diminution of maximum ideal stress in a tooth having $\alpha_{02} > \alpha_{01}$ is greater in terms of percent than the corresponding increase in weight.

Fig. 1 illustrates trends in the presence of identical modules and of $\alpha_{01}$ for $\Delta\sigma_{imax}$ % and for $\Delta G\% = [(G_a - G_0)/G_0]100$ where $\alpha_{02} = \Delta\alpha_0 + \alpha_{01}$.

When $\alpha_{02} = 30°$ (that is for $\Delta\alpha_0 = \alpha_{02} - \alpha_{01} = 10°$), $\Delta\sigma_{imax}$ % = –15.2 over an increase in $\Delta G\%$ weight = 2.54; therefore only a slight increase in weight is required to obtain a considerable reduction of the stress field at the tooth root.

On the basis of Fig. 1 we could see that maximum ideal stress being equal for both the symmetrical and asymmetrical teeth, it was possible to create asymmetrical $\alpha_{02} > \alpha_{01}$ teeth far lighter than traditional ones.

It was therefore important to study the possibility of obtaining, at identical maximum ideal stress rates, asymmetrical teeth smaller (smaller module) and lighter than corresponding symmetrical ones.

The possibility of quickly and precisely calculating smaller $m_{0a}$ modules having $\alpha_{02} > \alpha_{01}$ tooth and maximum ideal stress equal to that of a conventionally profiled tooth, was then tested under the same load conditions, that is, applying the same quantity of force.

This second state was devoted to seeking a calculation system capable of establishing $m_{0a}$ modules having $\alpha_{02} > \alpha_{01}$ in the presence of identical maximum ideal stress and identical load.

On the basis of the numerous cases studied, it appeared possible to calculate with a high degree of accuracy asymmetrical tooth $m_{0a}$ modules. This was done by departing from the conventional tooth $m_0$ module value. This $m_0$ module value is reduced according to the relationship described in Ref. 1 by a percentage value equal to the reduction of the $\Delta\sigma_{imax}$ % value. It is, therefore, possible to apply, with a high degree of accuracy, the following relational formula:

$$\Delta m_0\% = \Delta\sigma_{imax}\ \%$$

where

$$\Delta m_0\ \% = (m_{0a} - m_0)\ 100/m_0\ .$$

As $\Delta\sigma_{imax}$ % can be unequivocally established on the basis of a chosen $\Delta\alpha_0$ value, it is thus possible to

calculate, according to the relationship obtained, the value for $m_{0a}$, when the $m_0$ reference module is known.

The calculation methodology developed required the introduction of an index, definable as the $I_a$ "asymmetry index," which can be calculated on the basis of the following simple relational equation:

$$I_a = \psi\Delta\alpha_0 - \zeta\Delta\alpha_0^2 \quad (2)$$

The $I_a$ index allows for ready calculation of reduced $m_{oa}$ asymmetrical $\alpha_{02} > \alpha_{01}$ tooth modules, departing from the conventional tooth $m_0$ module:

$$m_{0a} = m_0\,(1 + I_a / 100) \quad (3)$$

where $I_a$ is derived from Eq. 2.

Eq. 3 was also tested by carrying out numerous comparisons between traditional and asymmetrical teeth. For every pair of teeth—one symmetrical and one asymmetrical tooth based on Eq. 3—the maximum ideal stress values, calculated using automated structural analysis, were compared. The difference between these maximum ideal stress values always stood at about 1% and never topped 2%.

Summing up, at the present stage of development, in order to calculate the size of an asymmetrical tooth, one may proceed as follows. It is possible to begin from the size of a traditional symmetrical tooth. Using the usual calculation procedures, the $m_0$ module is estimated. Then one must choose a value for $\alpha_{02} > \alpha_{01}$ compatible with the final geometrical characteristics desired.

By applying Eq. 3, it is possible to calculate the $m_{0a}$ module value for the asymmetrical tooth.

The above applies in the presence of equal force loads being brought to bear upon the symmetrical and asymmetrical teeth.

At this point it may be of interest to verify the degree of weight reduction applicable on the basis of the reduced module obtained by applying $\alpha_{02} > \alpha_{01}$.

In the graphs in Fig. 2 the values for ΔG% related to $\alpha_{02}$, for identical $\alpha_{01}$ values, and for maximum ideal stress at a constant within the ranges described above (±2%), this in the following two cases: $\alpha_{01}$ = 17°30' and $\alpha_{01}$ = 20°.

In the case of $\alpha_{01}$ = 17°30', for $\alpha_{02}$ = 30° (that is $\Delta\alpha_0$ = 12°30'), the weight

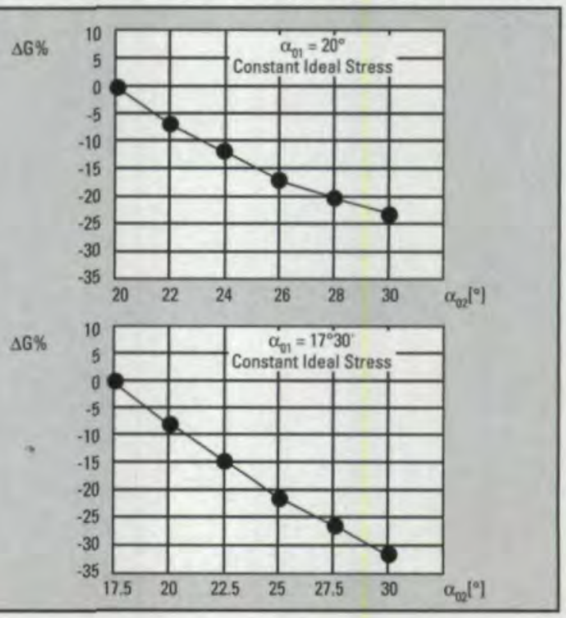

Fig. 2

Fig. 3

reduction obtained is over 30%; in this particular case the $\sigma_{imax\ a}$ value was 1% lower than that for the symmetrical tooth.

In Fig. 3, a traditional and an asymmetrical tooth in identical maximum ideal stress condition are shown. For both $\alpha_{01} = 17°30'$, the number of teeth is 20; furthermore $m_0$ = 2 mm, $m_{0a}$ = 1.63 mm.

It emerges from this figure that, in the presence of identical maximum ideal stress, it is more economical to use asymmetrical rather than traditional symmetrical teeth.

### Conclusions

In the light of these considerations it appears evident that, in cases where weight reduction is the primary need, the application of $\alpha_{02} > \alpha_{01}$ teeth leads to noteworthy results and shows the undeniable benefits of adopting teeth with non-conventional profiles.

At this stage the structural analysis and the entire study might stop here, were it not for the fact that the reduction of the module (adopted to achieve the same state of stress in the presence of identical load) involves a corresponding reduction of the diameter of the gear wheel, a reduction which, under conditions of identical normal force, leads to a reduction of the transmitted torque. The two gears, conventional and asymmetrical, while equivalent from a structural point of view, are not from the point of view of the transmitted torques. It is interesting to note how the reduction of the transmitted torque, which, as is known, is equal in terms of percent to the reduction of the tooth module, is also lighter in weight. It is therefore possible to create teeth having $\alpha_{02} > \alpha_{01}$ and smaller modules, capable of transmitting the same torque as traditional teeth, although their size is smaller. With this in mind the authors are carrying out a specific study for the calculation of $m_{0a}$ modules having the same transmitted torques. ✪

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